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Sensor-Based Bayesian Inference and Placement: Review and Examples

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Abstract: Systems, complex or otherwise, can be monitored through sensors placed at various functional levels to infer information about system reliability parameters (and by extension, reliability characteristics). Sensor placement directly affects the quantity and utility of inferred information, and need to be judiciously located throughout the relevant system. Sensors can be embedded or attached upon components, sub-systems or the entire system itself. Functional sensors can detect levels of functionality (including levels of degraded performance) and time to failure of the elements of the system they are monitoring. Data gathered from multiple system sensors will be 'overlapping' in that they are drawn from the same process or system at the same time. Overlapping data requires specific consideration for subsequent inference – system states observed by all sensors contextualize the data of all others. This paper is a review of how overlapping sensor data is analyzed in a Bayesian framework, and form part of a sensor placement optimization process to maximize information. This is particularly useful in scenarios where sensors are expensive to install with various resource constraints (such as volume and weight) limiting their use. The paper also presents review of a method of measuring the information utility of various sensor placement arrangements in a Bayesian construct of both on-demand and time-based continuous systems. Prior information is used to simulate evidence sets, which are then used to simulate posterior distribution of reliability metrics of interest. Information utility is derived from these posterior distributions, and an expected information utility is then attributed to sensor placement. Finally, examples of applying the methodologies discussed will be presented.

1 Introduction

Traditional reliability engineering involves analytical activities centered on components, followed by systemic performance prediction using a system model. This approach dominates contemporary reliability engineering practice, largely due to the 'diffuse' nature of system level data inference. Information inferred from system level data has to propagate across all of its sub-systems or components. This drives us to conduct analysis at the sub-system or component level either prior to or separate from systemic operation.

There are several, increasingly relevant, motivations for revisiting the extent to which we use system sensor data to inform reliability analysis. Firstly, sensors are becoming increasingly sophisticated and inexpensive. Secondly, testing is generally expensive and not always representative of operational conditions. Thirdly, sophisticated techniques have been developed to combine data from multiple sensors to yield information lost when considered in isolation. Fourthly, computational power is increasing. This allows reliability engineers to reevaluate their approaches to data gathering and analysis to better utilize data that would otherwise not be gathered during operation.

This paper reviews how system multi-sensor data can be subject to Bayesian inference to update our understanding of sub-system and component reliability parameters. It also reviews a methodology on how sensor placement can be optimized with multiple

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objectives, including the utility of inferred reliability information. The motivation to correctly analyze system failure data is clear insofar as having a better (or correct) understanding of component reliability characteristics better informs reliability, asset and maintenance management.

2 Sensor-based Bayesian Inference

When initiating a sensor-based reliability analysis process, the reliability engineer needs to understand whether the data is *overlapping* or not. *Overlapping data* meet the following criteria:

- ***Simultaneity*** - the data sets occur at the same time. When sensors are embedded at multiple places on the same system, they gather data simultaneously. The performance of one element of a system contextualizes what can be inferred from the performance of the remainder. If a sensor informs us that a sub-system has failed, this changes the 'meaning' of system failure (or success).
- ***Correspondence*** - the data sets are resultant from the same system or process. This is an inherent characteristic when sensors are embedded on the same system.

Non-overlapping data are those that fail either of these criteria [1] [2] [3] [4] [5] [6] [7]. This distinction is crucial, as it drastically changes the information that can be inferred. A simple example is a series system, where a sensor detects a sub-system failure. The system level sensor will always detect failure when the sub-system sensor detects failure regardless of how the other sub-systems function. In essence, the system level sensor yields no information when the sub-system sensor detects a failure. The only information that can be inferred when a sub-system sensor detects failure is that the sub-system in question has failed. This inference cannot be expanded upon by considering the state detected by the system level sensor, and hence of itself contains no additional information. If the data is treated as non-overlapping, the system level data incorrectly attributes failure evidence (in part) to sub-systems with no known relationship with system failure.

This example introduces the concept of *hierarchy*. Hierarchy is determined through *inference diagrams*. Sensor data is contextualized by *inferentially subordinate lower-level* sensor information. Sensor data contextualizes *immediately superior higher-level* sensor information. This relationship is illustrated in the inference diagram shown in Figure 1.

Non-overlapping data analysis is simpler and less computationally intensive. Accordingly, there is a tendency for *overlapping data* to be incorrectly analyzed as if it were *non-overlapping*. *Non-overlapping data* analysis involves the logical system 'deconstruction' to the elements (or sub-system) that each sensor detects functionality of. Each sensor detects 'top-event' functionality for its corresponding sub-system in a way that is completely independent of all other sensors. Hamada *et al*. have developed generalized methodologies for this approach for binary-state systems [8] and Graves *et al*. for multi-state systems [9].

Overlapping reliability data analysis also involves systemic deconstruction, but in this case all inferentially subordinate sensors must be incorporated as illustrated in Figure 1. Jackson and Mosleh have developed generalized methodologies for binary-state on-demand systems, [1] [3] [5] multi-state on-demand systems, [4] [5] and systems based on continuous life metrics (such as time) [2] [5]. The Jackson and Mosleh methodologies for the Bayesian analysis of *overlapping* sensor data from on-demand systems are computationally intensive and require a more involved computational algorithm [6].

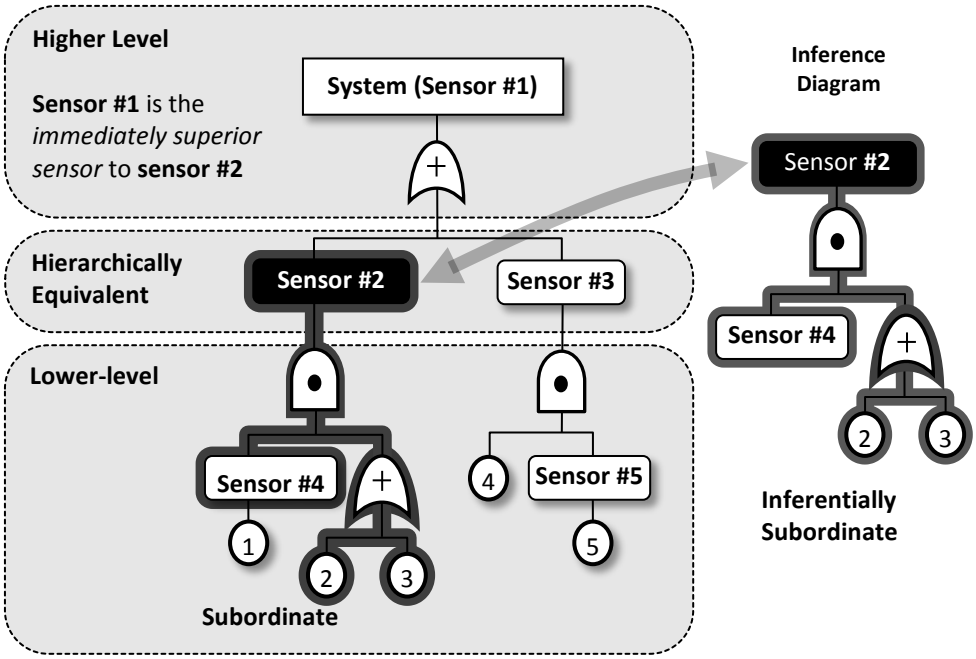

Figure 1: Inference Diagram of a Five Sensor, Five-Component System [2]

The differences between *overlapping* and *non-overlapping* reliability data, and the results they generate, have been illustrated in the following example. Consider a simple power module with two different and parallel transformer-filter sub-systems monitored by two sensors. Sensor #1 detects systemic failure. Sensor #2 detects Sub-system A failure.

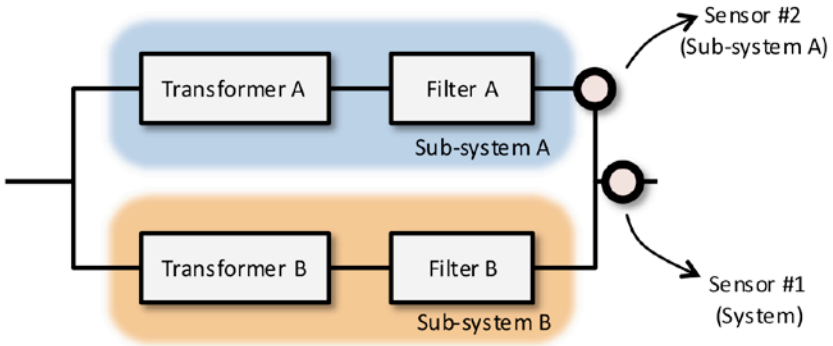

Figure 2: Reliability Block Diagram of a Power Module

Constant failure rates are assumed for each sub-system. The priori understanding of these failure rates are defined by uniform distributions of the sub-system mean time to failure, μ_A and μ_B respectively, across [0,20] expressed by Equation (1).

$$\boldsymbol{\theta} = \{\mu_A, \mu_B\} \tag{1}$$

where $\boldsymbol{\theta}$ is the set of all system reliability parameters.

The respective time to failure *Probability Density Functions* (PDFs) and *Cumulative Distribution Functions* (CDFs) for the system and sub-systems are $f(t|\boldsymbol{\theta})$, $f_A(t|\mu_A)$, $f_B(t|\mu_B)$ and $F(t|\boldsymbol{\theta})$, $F_A(t|\mu_A)$, $F_B(t|\mu_B)$, respectively. The system CDF and PDF are functions of sub-system CDFs and PDFs as follows:

$$F(t|\boldsymbol{\theta}) = F_A(t|\mu_A) \cdot F_B(t|\mu_B) \tag{2}$$

$$f(t|\boldsymbol{\theta}) = f_A(t|\mu_A) \cdot F_B(t|\mu_B) + f_B(t|\mu_B) \cdot F_A(t|\mu_A) \tag{3}$$

Suppose the following two evidence or data sets are gathered from each sensor (in terms of time to failure detection) from three tests:

$$E_1 = \tilde{\boldsymbol{t}}_1^S = \{11.12\,, 2.25\,, 6.99\} \tag{4}$$

$$E_2 = \tilde{\boldsymbol{t}}_2^S = \{11.12\,, 2.23\,, 6.99\} \tag{5}$$

where E_1 shows the data observed failure times from sensor 1 and E_2 shows the data from sensor 2.

Bayes' Theorem for this case may be expressed by

$$\pi_1(\boldsymbol{\theta}|E) = \frac{L(E|\boldsymbol{\theta})\pi_0(\boldsymbol{\theta})}{\int_{\forall\boldsymbol{\theta}} L(E|\boldsymbol{\theta}')\pi_0(\boldsymbol{\theta}')d\boldsymbol{\theta}'} \tag{6}$$

where $L(E|\boldsymbol{\theta})$ is the likelihood of observing evidence set E for given parameter set $\boldsymbol{\theta}$, and $\pi_0(\boldsymbol{\theta})$ is the joint probability density function that represents our prior understanding of $\boldsymbol{\theta}$, and $\pi_1(\boldsymbol{\theta})$ is the joint probability density function that represents the posterior inference of $\boldsymbol{\theta}$.

Firstly, the data sets in Equations (4) and (5) are treated as non-overlapping. This implies that the system is being tested in isolation to the testing of sub-system A. The system and sub-system A are each tested three times – a total of six separate tests. The likelihood function of observing the evidence sets is based on Equations (2) and (4):

$$L(E|\boldsymbol{\theta}) = L(\tilde{\boldsymbol{t}}_1^S, \tilde{\boldsymbol{t}}_2^S|\mu_A, \mu_B) \tag{7}$$

$$= \prod_{\forall t \in \tilde{\boldsymbol{t}}_1^S} [f_A(t|\mu_A) \cdot F_B(t|\mu_B) + f_B(t|\mu_B) \cdot F_A(t|\mu_A)] \times \prod_{\forall t \in \tilde{\boldsymbol{t}}_2^S} f_A(t|\mu_A)$$

$$= \prod_{\forall t \in \tilde{\boldsymbol{t}}_1^S} \left[\frac{e^{-\frac{t}{\mu_A}}}{\mu_A} \cdot \left(1 - e^{-\frac{t}{\mu_B}}\right) + \frac{e^{-\frac{t}{\mu_B}}}{\mu_B} \cdot \left(1 - e^{-\frac{t}{\mu_A}}\right) \right] \times \prod_{\forall t \in \tilde{\boldsymbol{t}}_2^S} \frac{e^{-\frac{t}{\mu_A}}}{\mu_A}$$

Secondly, the data sets in Equations (4) and (5) are treated as overlapping. This involves each element of the data set constituting an individual test. As there are three evidence or data elements, there are three tests that comprise of a pair of times to failure detection. Each pair consists of times that the sensor level and sub-system A sensors detect failure. This first test involves both sensors detecting failure at the same time (11.12 months). This implies that sub-system B must have failed prior to 11.12 months, with the failure of sub-system A immediately triggering systemic failure. For the second test, the sub-system A sensor detected failure prior to the system sensor. This implies that sub-system A failed (at 2.23 months), prior to sub-system B and the entire system failing (at 2.25 months). The same scenario as that observed in the first test was observed in the third test, with sub-system B failing before sub-system A did, at 6.99 months.

The resultant likelihood functions of the sub-system mean times to failure are illustrated in Figure 3. The non-overlapping data posterior distribution was generated using Equation (7). The overlapping data posterior distribution was generated from equations that will be developed subsequently in this paper. The most marked difference

is the effect that different analysis methodologies had on the information inferred about sub-system B. The maximum likelihood estimate (MLE) of the sub-system B mean time to failure (μ_B) is 3.98 and 1.99 months when the data is assumed to be non-overlapping and overlapping respectively. The corresponding results for sub-system A mean time to failure (μ_A) are less divergent at 5.99 and 6.78 months.

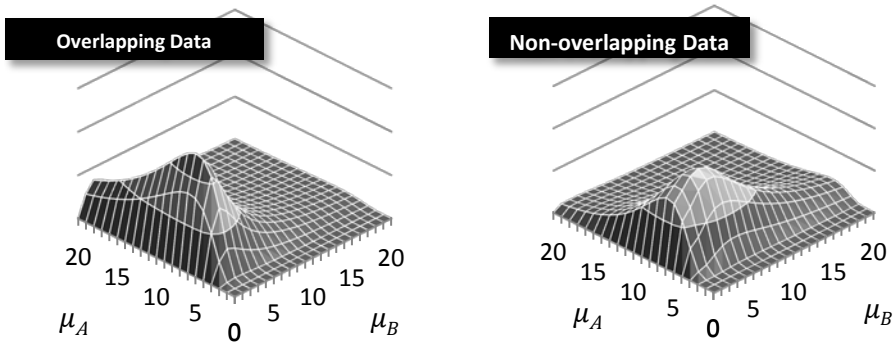

Figure 3: Posterior Distribution of Power Unit Sub-system Mean Times to Failure

The inaccuracies associated with incorrectly analyzing data as non-overlapping when it is in fact overlapping (or vice versa) are noticeable. The average of the sub-system A failure detection times is 6.78 months, which is the overlapping data MLE. However, the non-overlapping data MLE is lower as the system level evidence set is inferring incorrect information about sub-system A's reliability characteristics. This is exacerbated when considering sub-system B. Two of the tests involve the entire system failing when sub-system A fails, meaning that subs-system B has failed sometime earlier. The overlapping data likelihood takes this into account with a substantially lower sub-system B mean time between failures MLE. This is further demonstrated when estimating the system mean time between failures from sub-system mean time between failure MLEs. These estimates are 6.49 and 8.25 for the overlapping and non-overlapping data likelihood functions, respectively. The figure of 6.49 aligns more closely with evidence set 1, while the figure of 8.25 is significantly disparate.

2.1 Non-overlapping Sensor Data for On-demand Systems

The state of knowledge for an on-demand system is defined by the joint probability distribution of the component state probabilities. Most components have two states denoting functionality or failure (*i.e.*, binary state systems). It is increasingly common for multiple states of degradation to be incorporated into reliability analysis. For a system with z distinct states, the state '0' represents complete functionality and the state 'z – 1' represents failure. All intermediate states represent varying degrees of degradation. Consider the following representations for a system with n components:

p_j^x as the probability of the j^{th} component being in the x^{th} state noting that $0 \leq x \leq z$ (8)

$\tilde{\boldsymbol{p}}_j = \{p_j^1, p_j^2, p_j^3, \dots, p_j^{z-1}\}$ as the vector of all state probabilities of the j^{th} component, (9)

$\mathbf{p} = \{\tilde{\boldsymbol{p}}_1, \tilde{\boldsymbol{p}}_2, \tilde{\boldsymbol{p}}_3, \dots, \tilde{\boldsymbol{p}}_j, \dots, \tilde{\boldsymbol{p}}_n\}$ as the $(z-1) \times n$ matrix of component type state probabilities for the entire system, and (10)

$\pi_0(\mathbf{p})$ as the prior distribution of the unknowns on interest that define the state of knowledge for a system (11)

The state probability for functionality, p_j^0, is omitted from the vector in Equation (9). As all other state probabilities sum to 1, defining them also defines p_j^0. This means that you need $(z-1)$ probabilities to describe the state of knowledge. For a system with m sensors, evidence will be gathered in the form outlined in Table 1 where $k_{S_i}^x$ represents the number of times the i^{th} sensor detected the x^{th} state.

Table 1: Expression of $(z-1)$ State of 'Demand on Standby' System with m Sensors.

System	State No	S_1	S_2	…	S_m
Number of observed system states	0	$k_{S_1}^0$	$k_{S_2}^0$	…	$k_{S_m}^0$
	1	$k_{S_1}^1$	$k_{S_2}^1$	…	$k_{S_m}^1$
	…	…	…	…	…
	z - 1	$k_{S_1}^{z-1}$	$k_{S_2}^{z-1}$	…	$k_{S_m}^{z-1}$
Number of demands		r_{S_1}	r_{S_2}	…	r_{S_m}

The corresponding likelihood function, based on the multinomial distribution, is defined below:

$$L(E|\mathbf{p}) = \prod_{i=1}^{m}\left[r_{S_i}! \prod_{x=0}^{z-1} \frac{\left(p_{S_i}^x\right)^{k_{S_i}^x}}{k_{S_i}^x!}\right] \propto \prod_{i=1;x=0}^{m;z-1} \left(p_{S_i}^x\right)^{k_{S_i}^x} \quad (12)$$

where $p_{S_i}^x$ is the probability of the i^{th} sensor detecting the x^{th} state, expressed as a function of all subordinate component state probabilities [9].

2.2 Overlapping Sensor Data for On-demand Systems

Analysis of overlapping data is considerably more complicated than analysis of non-overlapping data. Overlapping data is of the same form as that in Table 1 (*i.e.*, the number of demands obviously remain consistent for all sensors). The likelihood function becomes:

$$L(E|\boldsymbol{p}) \propto \sum_{\forall \vec{v}_a \in \mathbf{v}_E} \left[\prod_{l=1}^{z^n} \frac{1}{(v_l)_a!} \left(\prod_{b=1}^{n} p_b^{(x_b)_l}\right)^{(v_l)_a}\right] \quad (13)$$

where $p_b^{(x_b)_l}$ is the probability of the b^{th} component being in the $(x_b)_l^{th}$ state with the remainder of the terms explained in greater detail below. [4]

Every time an on-demand system is tested (or subject to a demand), it will respond by adopting a particular state vector - $\tilde{\boldsymbol{x}}$. A state vector defines the state of each component of the system. The l^{th} state vector, $\tilde{\boldsymbol{x}}_l$, is defined in Equation (14).

$$\tilde{\boldsymbol{x}}_l = \{(x_1)_l, (x_2)_l, (x_3)_l, \dots, (x_b)_l, \dots, (x_n)_l\} \quad (14)$$

where $(x_b)_l$ is the state variable of the b^{th} component (as represented in the l^{th} state vector), and n is the number components.

When subject to r demands, system behavior will be described in r state vectors. Each particular combination is defined by a vector $\tilde{\boldsymbol{v}}$, whose elements describe the number of

times a particular state vector appears in that combination. The vector $\tilde{\boldsymbol{v}}_a$ defines the a^{th} combination of r state vectors as defined in Equation (15).

$$\tilde{\boldsymbol{v}}_a = \{(v_1)_a, (v_2)_a, \dots, (v_l)_a, \dots, (v_{z^n})_a\} \quad (15)$$

where, $(v_l)_a$ is the number of occurrences of the l^{th} state vector in $\tilde{\boldsymbol{v}}_a$, noting that $\sum_{l=1}^{z^n} v_l = r$.

The set $\mathbf{v}_E$ contains all vectors $\tilde{\boldsymbol{v}}$ that define state vector combinations that imply the evidence. Any test regime with r demands may have a number of different state vector combinations that correlate with observed numbers of failure detected at various sensors.

$$\tilde{\boldsymbol{v}}_a \in \mathbf{v}_E \text{ iff } \tilde{\boldsymbol{v}}_a \rightarrow E \text{ (implies the evidence)} \quad (16)$$

The majority of computational effort is dedicated to constructing the set $\mathbf{v}_E$. The first step involves establishing structure functions for the system. The structure function for the i^{th} sensor, ϕ_i^S, defines the state detected by the i^{th} sensor for a given component state vector.

$$\phi_i^S(\tilde{\boldsymbol{x}}) = \phi_i^S(\{x_1, x_2, x_3, \dots, x_b, \dots, x_n,\}) \quad (17)$$

$$= \begin{cases} z-1 \dots if\ the\ i^{th}\ (sub-)system\ has\ completely\ failed \\ \dots \\ 1 \dots if\ the\ i^{th}\ (sub-)system\ is\ in\ the\ first\ degraded\ state \\ 0 \dots if\ the\ i^{th}\ (sub-)system\ is\ completely\ functional \end{cases}$$

The number of x^{th} states observed by the i^{th} sensor for the a^{th} combination of state vectors (defined by $\tilde{\boldsymbol{v}}_a$) is given by $k_i^{S(x)}\big|_{\tilde{\boldsymbol{v}}_a}$ in Equation (18).

$$k_i^{S(x)}\big|_{\tilde{\boldsymbol{v}}_a} = \sum_{l=1}^{z^n} (v_l)_a \times \begin{pmatrix} 1\ if\ \phi_i^S(\tilde{\boldsymbol{x}}_l) = x \\ 0\ if\ \phi_i^S(\tilde{\boldsymbol{x}}_l) \neq x \end{pmatrix} \quad (18)$$

The evidence failure combination function is a function that determines if the a^{th} combination of state vectors (defined by $\tilde{\boldsymbol{v}}_a$) implies the evidence set $E = \{\tilde{\boldsymbol{k}}_1^S, \tilde{\boldsymbol{k}}_2^S, \dots, \tilde{\boldsymbol{k}}_m^S, r\}$. For this to occur, the number of states detected by each sensor implied by $\tilde{\boldsymbol{v}}_a$ must equal the number contained in the evidence set.

$$\tilde{\boldsymbol{v}}_a \in \mathbf{v}_E \quad \text{iff} \quad k_i^{S(x)}\big|_{\tilde{\boldsymbol{v}}_a} = \sum_{l=1}^{z^n} (v_l)_a \times \begin{pmatrix} 1\ if\ \phi_i^S(\tilde{\boldsymbol{x}}_l) = x \\ 0\ if\ \phi_i^S(\tilde{\boldsymbol{x}}_l) \neq x \end{pmatrix} \dots \forall i \in 1,2,3, \dots, m \quad (19)$$

Finding the set of all $\tilde{\boldsymbol{v}}_a$ that satisfy Equation (19) increases in complexity with the size or the number of states of the systems. See the algorithm for establishing the set $\mathbf{v}_E$ in [6].

2.3 Overlapping Sensor Data for Time-based (Continuous) Systems

Time-based systems require a different approach to overlapping data analysis [2]. Each component in a time-based system has a set of parameters that defines its time-based reliability characteristics in lieu of the component state probabilities. The unknowns of interest become the set of these parameters for each component.

$\tilde{\boldsymbol{\theta}}_b$ is the vector of all time based reliability characteristics for the b^{th} component, allowing component time to failure CDF and PDF to be expressed as $F_b(t|\tilde{\boldsymbol{\theta}}_b)$ and $f_b(t|\tilde{\boldsymbol{\theta}}_b)$ respectively (20)

The set of reliability parameters of the b^{th} component, $\tilde{\boldsymbol{\theta}}_b$, defines the probability of component having failed at time t (or the *CDF*). The set of all n component parameter sets (and the unknown of interest) for the system is:

$$\boldsymbol{\theta} = \{\widetilde{\boldsymbol{\theta}}_1, \widetilde{\boldsymbol{\theta}}_2, \widetilde{\boldsymbol{\theta}}_3, \dots, \widetilde{\boldsymbol{\theta}}_b, \dots, \widetilde{\boldsymbol{\theta}}_n\} \quad (21)$$

The evidence is the vector of m sensors' failure detection times.

$$\tilde{\boldsymbol{t}}^S = \{t_1^S, t_2^S, t_3^S, \dots, t_i^S, \dots, t_m^S\} \quad (22)$$

where, t_i^S is the time failure is detected by the i^{th} sensor

There will be a logical relationship (such as that represented by a fault tree) between the probabilities of failure at any level in the system and those of subordinate components or sub-systems. For example, by establishing cut-sets and using Boolean this can be determined. The states detected by sensors can be defined as:

Probability that the i^{th} sensor has detected failure at time t = (23)

$$F_i^S\left[\dots, \hat{F}_{i'}^S\big|_t, \dots; \dots, F_{b'}\left(t|\widetilde{\boldsymbol{\theta}}_{b'}\right), \dots\right]$$

where, $\hat{F}_{i'}^S\big|_t$ is the *apparent failure detection probability* of the i'^{th} inferentially subordinate sensor (defined below), $i' \in \boldsymbol{i}_i^{\subset S}$ where $\boldsymbol{i}_i^{\subset S}$ is the set of indices of all sensors that are *inferentially subordinate* to the i^{th} sensor, and $b' \in \boldsymbol{b}_i^{\subset S}$ where $\boldsymbol{b}_i^{\subset S}$ is the set of indices of all components that are *inferentially subordinate* to the i^{th} sensor

Equation (23) expresses 'higher-level' system failure probabilities in terms of 'lower-level' failure probabilities. By extension, Equation (23) can be expressed in terms of inferentially subordinate sensor failure detection times, and the parameter sets of inferentially subordinate components.

$$F_i^S\left[\dots, \hat{F}_{i'}^S\big|_t, \dots; \dots, F_{b'}\left(t|\widetilde{\boldsymbol{\theta}}_{b'}\right), \dots\right] = F_i^S(t|\boldsymbol{\theta}_i^{\subset S}, \tilde{\boldsymbol{t}}_i^{\subset S}) \quad (24)$$

where $\boldsymbol{\theta}_i^{\subset S}$ is the set of parameter sets of component *inferentially subordinate* to the i^{th} sensor $\left(\boldsymbol{\theta}_i^{\subset S} = \{\dots, \widetilde{\boldsymbol{\theta}}_{b'}, \dots\}\right)$, $\tilde{\boldsymbol{t}}_i^{\subset S}$ is the set of failure detection times of sensors *inferentially subordinate* to the i^{th} sensor $(\tilde{\boldsymbol{t}}_i^{\subset S} = \{\dots, t_{i'}, \dots\})$

Incorporating the overlapping nature of the data, the way in which in *inferentially subordinate* sensor failure detection probabilities, interact with *inferentially superior* sensor failure detection probabilities must be incorporated to contextualize the evidence. This is affected by using *the apparent failure detection probabilities* (CDF and PDFs) of inferentially subordinate sensors that incorporate inferentially subordinate evidence.

The i'^{th} inferentially subordinate sensor (in this scenario) either detect failure or success up to a particular point in time, $t_{i'}^S$. The latter scenario is analogous to right censored data. Should the i'^{th} inferentially subordinate sensor be right censored, then $t_{i'}^S \in \tilde{\boldsymbol{t}}_{cens}^S$ where $\tilde{\boldsymbol{t}}_{cens}^S$ is the set of all right-censored sensor detection times. These relationships are formally incorporated by Equation (25), which describes the *apparent failure detection probability or CDF* from the perspective of the inferentially superior sensor.

$$\hat{F}_{i'}^S(t|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}) = \begin{cases} 0 & \dots \quad if\ t_{i'}^S \in \tilde{\boldsymbol{t}}_{cens}^S\ and\ t \le t_{i'}^S \\ \frac{F_{i'}^S\left(t|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}\right) - F_{i'}^S\left(t_{i'}^S|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}\right)}{1 - F_{i'}^S\left(t_{i'}^S|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}\right)} & \dots \quad if\ t_{i'}^S \in \tilde{\boldsymbol{t}}_{cens}^S\ and\ t > t_{i'}^S \\ H(t - t_{i'}^S) & \dots \quad otherwise \end{cases} \quad (25)$$

where, $H(t)$ is the *unit* or *Heaviside step function*

The resulting likelihood function is written in Equation (26)

$$L(\tilde{\boldsymbol{t}}^S|\boldsymbol{\theta}) = L\left(\{t_1^S, \dots, t_m^S\}|\widetilde{\boldsymbol{\theta}}_1, \dots, \widetilde{\boldsymbol{\theta}}_n\right) = \prod_{i=1}^{m} L_i^S(t_i^S|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}) \tag{26}$$

$$\propto \prod_{i=1}^{m} \begin{bmatrix} 1 - F_i^S(t_i^S|\boldsymbol{\theta}_i^{\subset S}, \tilde{\boldsymbol{t}}_i^{\subset S}) & \dots \quad \forall t_i^S \in \tilde{\boldsymbol{t}}_{cens}^S \\ \left.\frac{\partial F_i^S\left(t|\boldsymbol{\theta}_i^{\subset S}, \tilde{\boldsymbol{t}}_i^{\subset S}\right)}{\partial \bar{F}_{i'}^S\left(t|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}\right)}\right|_{t=t_i^S} & \dots \quad if\ t_i^S = t_{i'}^S \\ \left.\sum_{\forall b' \in \boldsymbol{b}_i^{\subset S}} \left(\frac{\partial F_i^S\left(t|\boldsymbol{\theta}_i^{\subset S}, \tilde{\boldsymbol{t}}_i^{\subset S}\right)}{\partial F_b\left(t|\widetilde{\boldsymbol{\theta}}_b\right)} \cdot f_b\left(t|\widetilde{\boldsymbol{\theta}}_b\right)\right)\right|_{t=t_i^S} & \dots \quad otherwise \end{bmatrix}$$

The case where $t_i^S = t_{i'}^S$ implies that the i^{th} sensor detected failure at the same time that the i'^{th} inferentially subordinate sensor detected failure. Also it implies that the i^{th} sensor detected failure *because* of the failure detected by the i'^{th} inferentially subordinate failure.

Equation (26) is complicated, and its inclusion of partial derivatives may make some systemic scenarios difficult to solve. It can be greatly simplified should the 'real world' inaccuracies associated with time measurement devices be incorporated. If the timepiece is (for example) accurate to one hundredth of a second, one can only ever say with certainty that failure was detected inside a time interval of one hundredth of a second. In this case the evidence becomes a vector of failure detection time estimates.

$$\hat{\tilde{\boldsymbol{t}}}^S = \{\hat{t}_1, \hat{t}_2, \hat{t}_3, \dots, \hat{t}_i, \dots, \hat{t}_m|\Delta t_1, \Delta t_2, \Delta t_3, \dots, \Delta t_i, , \dots, \Delta t_m\}$$
where $\hat{t}_i$ is the estimated time of failure detection by the i^{th} sensor with accuracy Δt_i (27)

The likelihood function in Equation (26) is modified to the simpler likelihood function in Equation (28) to incorporate measurement inaccuracy [5].

$$L\left(\hat{\tilde{\boldsymbol{t}}}^S|\boldsymbol{\theta}, \Delta t\right) = L\left(\{\hat{t}_1^S, \dots, \hat{t}_m^S\}|\{\widetilde{\boldsymbol{\theta}}_1, \dots, \widetilde{\boldsymbol{\theta}}_n\}, \{\Delta t_1, \dots, \Delta t_m\}\right) = \prod_{i=1}^{m} [F_i^S(\hat{t}_i^S + \Delta t_i|\boldsymbol{\theta}_i^{\subset S}, \tilde{\boldsymbol{t}}_i^{\subset S}) - F_i^S(\hat{t}_i^S|\boldsymbol{\theta}_i^{\subset S}, \tilde{\boldsymbol{t}}_i^{\subset S})] \tag{28}$$

The *apparent failure detection probability* from the perspective of the inferentially superior sensors defined in Equation (25) may be amended for measurement inaccuracy.

$$\hat{F}_{i'}^S\left(t|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}, \Delta t_{i'}\right) = \begin{cases} 0 & t \le t_{i'}^S \\ \frac{F_{i'}^S\left(t|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}\right) - F_{i'}^S\left(\hat{t}_{i'}^S|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}\right)}{1 - F_{i'}^S\left(\hat{t}_{i'}^S|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}\right)} & t_{i'}^S \in \tilde{\boldsymbol{t}}_{cens}^S \text{ and } t > t_{i'}^S \\ \frac{F_{i'}^S\left(t|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}\right) - F_{i'}^S\left(\hat{t}_{i'}^S|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}\right)}{F_{i'}^S\left(\hat{t}_{i'}^S + \Delta t_{i'}|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}\right) - F_{i'}^S\left(\hat{t}_{i'}^S|\boldsymbol{\theta}_{i'}^{\subset S}, \tilde{\boldsymbol{t}}_{i'}^{\subset S}\right)} & t_{i'}^S < t \le t_{i'}^S + \Delta t_{i'} \\ 1 & otherwise \end{cases} \tag{29}$$

This represents a significant simplification when contrasted with Equation (26), as it does not involve partial derivatives. This also reflects the enduring reality that all time measurement devices have an inherent level of accuracy or precision, and can never ultimately establish the exact time of failure detection.

3 Sensor Placement and Information

The ultimate aim of this analysis is to improve the state of knowledge one has of a system or a process. Typically, there is no need to quantify this information: the evidence 'is what it is.' However, quantifying the expected 'amount' of information that can be gathered via different frameworks is beneficial if one can modify those frameworks – as is the case where one has a choice regarding sensor placement.

3.1 Information of Reliability Unknowns of Interest

Reliability 'information' is measured in terms of our understanding of the unknowns of interest - typically the joint probability distribution of component reliability parameters and variables. Popular information metrics are based on the normalized magnitude of the covariance of this joint probability distribution [10] [11]. These metrics imply that information is maximized when the magnitude of the covariance is minimized, which makes intuitive sense. Reducing the variation or uncertainty as much as possible increases the information one has of that random variable's true value. Two popular information metrics are listed below, noting that other metrics exist across multiple fields of research.

a. **Fisher Information.** The *Fisher Information* of a set of random variables with a joint probability distribution provides the lower bounds on variance and covariance. Maximizing *Fisher Information* therefore minimizes the variation of these random variables [10].
b. **Shannon Information.** Developed by Shannon [11], described by *Differential Entropy*, quantifies the uncertainty of a set of random variables with a joint probability distribution. Minimizing the *differential entropy* decreases the uncertainty, and hence increases information content.

Information can be quantified through an information utility function. A *utility function* quantifies the 'worth' or 'value' of a particular (typically experimental) scenario in a probabilistic Bayesian construct, and is designated with the symbol U. The information utility is expressed as a function of the physical rules of observation, ε [12]. For example, ε_1 could refer to a test methodology based on time to failure. Alternately, ε_2 could refer to another test methodology based on number of failures observed in an arbitrary interval. In the first framework, the evidence, E_1, will consist of failure times. In the second framework, E_2 will consist of the number of functional or surviving components, and the number of failed components. Both frameworks are examining the same phenomenon (the failure time of the components), but are gathering information in different ways. The likelihood function is commonly expressed as a conditional probability of E on the underlying unknowns of interest, $\boldsymbol{\theta}$. It is more completely expressed as conditional on ε as well. This modifies Bayes Theorem to the form expressed by Equation (30).

$$\pi_1(\boldsymbol{\theta}|E,\varepsilon) = \frac{L(E|\boldsymbol{\theta},\varepsilon)\pi_0(\boldsymbol{\theta})}{\int_{\forall\boldsymbol{\theta}} L(E|\boldsymbol{\theta}',\varepsilon)\pi_0(\boldsymbol{\theta}')d\boldsymbol{\theta}'} \tag{30}$$

The information utility, U^I, is a function of the joint probability distribution of the unknowns of interest that form the state of knowledge formally given by Equation (31).

$$\text{Information utility} = U^I[\pi(\boldsymbol{\theta})] \text{ or } U^I(\pi) \tag{31}$$

When considering how the information utility should be constructed, the 'principle' of information sought needs to be selected. Example principles are listed below.

a. **Information of Posterior Distribution.** The information of the *posterior distribution* represents the information produced by a testing framework. It is inherent upon the prior information and the information gained during the test.

b. **Information difference of Posterior/Prior distributions.** Many methodologies focus on the *information difference of posterior/prior distributions*, as this represents the information gained by the test methodology, ε [12].
c. **Information of specific parameters.** Information of certain parameters may be more valuable than the information of others. It maybe that the test involves a system or process with many parameters, but one parameter in particular is the most valuable in terms of the confidence of subsequent physical predictions.
d. **Information of variables that are functions of parameters.** It may be desirable to gain information about the reliability characteristics of a particular component as opposed to its underlying parameters.

Generally, we are interested in the information associated with the posterior distribution as this represents our updated state of knowledge. We can associate an information utility with the expected posterior distribution for each ε and identify the one that maximizes it. By extension, the information utility can be expressed in terms of both ε and E as follows:

$$U^I[E, \varepsilon|\pi_0(\boldsymbol{\theta})] \text{ or } U^I(E, \varepsilon|\pi_0) \text{ or } U^I(\pi_1|\pi_0) \quad (32)$$

noting that E, ε, and $\pi_0(\boldsymbol{\theta})$ defined the posterior distribution $\pi_1(\boldsymbol{\theta}|E, \varepsilon)$ using Bayes' Theorem in Equation (30)

Several examples of information based utility functions are included in Table 2.

Table 2: Examples of Information-Based Utility Functions of Experimental Frameworks [5]

Utility Function $U^I(E, \varepsilon\|\boldsymbol{\pi_0})$	Description
$\int_{\forall\theta} \ln[\pi_1(\theta'\|E,\varepsilon)]\,\pi_1(\theta'\|E,\varepsilon)d\theta' - \int_{\forall\theta} \ln[\pi_0(\theta')]\,\pi_0(\theta')d\theta'$	Expected improvement of *Shannon Information* by posterior distribution when compared to prior distribution **[12]**
$\min\left(\frac{1}{\sigma^2_\theta}\right)$ where $\sigma^2_\theta = \left\{\tilde{\sigma}^2_{\tilde{\theta}_1}, \tilde{\sigma}^2_{\tilde{\theta}_2}, \dots, \tilde{\sigma}^2_{\tilde{\theta}_n}\right\}$ and $\theta \sim \pi_1(\theta\|E,\varepsilon)$	Inverse of the largest posterior parameter variance
$\frac{1}{\sum_{j=1}^{n}\sigma^2_{\theta_j}}$ where $\theta \sim \pi_1(\theta\|E,\varepsilon)$	Inverse of the sum of the posterior variance of all parameters.

3.2 Bayesian Experimental Design

Bayesian experimental design deals with test scenarios where parameters such as test size and duration can be manipulated. The principles of Bayesian experimental design are equally valid for scenarios involving sensor placement. The experimental framework can also be denoted ε, as this defines the nature of evidence gathering.

Consider a hypothetical scenario where information is maximized and there is therefore no uncertainty with the parameter set $\boldsymbol{\theta}$. This implies that all values contained in the set $\{\tilde{\boldsymbol{\theta}}_1, \tilde{\boldsymbol{\theta}}_2, \tilde{\boldsymbol{\theta}}_3, \dots, \tilde{\boldsymbol{\theta}}_b, \dots, \tilde{\boldsymbol{\theta}}_n\}$ are known. Therefore:

$$\Pr(E|\boldsymbol{\theta}, \varepsilon) = \text{the probability of observing evidence set } E \text{ for parameter set } \boldsymbol{\theta} \text{ in experimental framework } \varepsilon. \quad (33)$$

In reality, there will always be a level of uncertainty with the parameter set $\boldsymbol{\theta}$. If this was not the case, then there would be no need to conduct any testing. The probability of observing a specific evidence set is listed in Equation (34).

$$\Pr[E|\varepsilon, \pi_0(\boldsymbol{\theta})] \text{ or } \Pr(E|\varepsilon, \pi_0) = \int_{\forall\boldsymbol{\theta}} \Pr(E|\varepsilon, \boldsymbol{\theta}')\pi_0(\boldsymbol{\theta}')d\boldsymbol{\theta}' \quad (34)$$

A key characteristic of Equation (34) is that the probability of observing specific evidence sets is conditional on ε. This allows the probabilities of specific evidence sets being calculated as a function of the test methodology. If E is the set of all possible evidence sets, and E_q is an element of this set, the following can be written:

$$f\left(U_q^I\right) = \Pr\left(U_q^I\right) = \Pr\left(E_q|\varepsilon, \pi_0\right) \tag{35}$$

where U_q^I or $U_q^I\left(E_q|\varepsilon, \pi_0\right)$ is the information utility of the q^{th} possible evidence set, E_q, and the test methodology, ε, given the prior distribution of unknowns of interest, $\pi_0(\boldsymbol{\theta})$

The expected information utility for ε is given in Equation (36).

$$\bar{U}^I(\varepsilon|\pi_0) = \int_{\forall q} U_q^I \cdot f\left(U_q^I\right) dU_q^I = \sum_{\forall q} U_q^I \cdot \Pr\left(E_q|\varepsilon, \pi_0\right) \tag{36}$$

Substituting Equation (35) into Equation (36) yields Equation (37).

$$\bar{U}^I(\varepsilon|\pi_0) = \sum_{\forall q} U_q^I \cdot \left[\int_{\forall \boldsymbol{\theta}} \Pr\left(E_q|\varepsilon, \boldsymbol{\theta}'\right)\pi_0(\boldsymbol{\theta}')d\boldsymbol{\theta}'\right] \tag{37}$$

The expected information utility is dependent on the information framework, and the type of information that the reliability engineer selects. Once one has done so, the expected posterior distribution of the unknowns of interest can help select the testing methodology (or sensor placement) that provides the most information.

3.3 Optimizing Information

Information optimization should be an element of a multi-objective optimization framework, where considerations including cost are incorporated. There is significant literature on multi-objective optimization [13], but information optimization only will be examined below in the context of sensor placement. There are eight steps in that process.

a. **Formalize Prior Information.** Prior information of a system is summarized in the prior distribution: $\pi_0(\boldsymbol{\theta})$ where, $\boldsymbol{\theta}$ is the set of *unknowns of interest* or *parameters* $\left\{\widetilde{\boldsymbol{\theta}}_1, \widetilde{\boldsymbol{\theta}}_2, \widetilde{\boldsymbol{\theta}}_3, \dots, \widetilde{\boldsymbol{\theta}}_n\right\}$ (38)

b. **Identify Available Sensor Locations.** All possible sensor locations should be identified. $\boldsymbol{\varepsilon}$ = set of possible sensor locations (39)
A set of possible sensor arrangements (noting that a single arrangement comprises 1 or more possible sensor locations) needs to be compiled:
$\boldsymbol{\varepsilon}' = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots\}$,where $\boldsymbol{\varepsilon}' \subseteq \boldsymbol{\varepsilon}$ and $\varepsilon_i \in \boldsymbol{\varepsilon}$ (40)

c. **Define Information Utility.** The information utility function, to be selected by the reliability engineer, will be a conditional function of the posterior distribution given the prior distribution of the unknowns of interest.
$U^I(E, \varepsilon|\pi_0)$ or $U^I(\pi_1|\pi_0)$ (41)

d. **Describe System Logic (Bayesian Analysis).** The system logic links observed system behavior (evidence) with its underlying parameters.

e. **Developing the Likelihood Function.** As discussed in the development of Equation (30), the likelihood function is conditional on systemic parameters (through system logic) and the testing methodology, ε: $L(E|\boldsymbol{\theta}, \varepsilon)$ (42)

f. **Simulation of evidence.** The possible evidence sets along with their probabilities need to be simulated from the prior distribution of unknowns of interest, $\pi_0(\boldsymbol{\theta})$. There are multiple approaches to this, but an approach utilizing Monte-Carlo simulation is outlined below. For on-demand systems:

1. *Sampling unknowns of interest* – **θ**. Monte-Carlo simulation can be used to randomly draw joint samples of the unknowns of interest.
2. *Deriving structure functions.* The structure function (as previously discussed) calculates how system state vectors, $\tilde{\boldsymbol{x}}$, influence what sensors detect. Compiling structure functions for each sensor allows the *sensor information vector* (SIV), $\tilde{\boldsymbol{x}}^S$, to be calculated.
   $$\tilde{\boldsymbol{x}}^S = \phi_\varepsilon(\tilde{\boldsymbol{x}}) \quad (43)$$
   where ϕ_ε is the *structure function* which is dependent on the test methodology, ε, $\tilde{\boldsymbol{x}}^S = \{x_1^S, x_2^S, \dots, x_i^S, \dots, x_m^S\}$ where x_i^S is the state detected by the i^{th} sensor, and $\tilde{\boldsymbol{x}} = \{x_1, x_2, \dots, x_j, \dots, x_n\}$ where x_j is the state of the i^{th} component
3. *Simulation of state vectors* - $\tilde{\boldsymbol{x}}$. Each randomly drawn set of the *unknowns of interest* defines component state probabilities. Monte-Carlo simulation can then be used to randomly draw *component state vectors*. The resultant set of *component state vectors* will then have an equal probability associated with each.
   $$\Pr(\tilde{\boldsymbol{x}}_l|\pi_0) \approx \frac{d_l}{d} \quad (44)$$
   where, d_l is the number of times the l^{th} component state vector, $\tilde{\boldsymbol{x}}_l$, is drawn from a total of d simulations
4. *Simulation of Sensor Information Vectors (SIV)* - $\tilde{\boldsymbol{x}}^S$. SIV, as defined in Equation (43), can now be simulated based on the sampling of component state vectors. The probability of each SIV is:
   $$Pr(\tilde{\boldsymbol{x}}_{l'}^S|\varepsilon,\pi_0) = \sum_{\forall \tilde{\boldsymbol{x}}_l : \tilde{\boldsymbol{x}}_{l'}^S = \phi_\varepsilon(\tilde{\boldsymbol{x}}_l)} \Pr(\tilde{\boldsymbol{x}}_l|\pi_0) \approx \sum_{\forall \tilde{\boldsymbol{x}}_l : \tilde{\boldsymbol{x}}_{l'}^S = \phi_\varepsilon(\tilde{\boldsymbol{x}}_l)} \frac{d_l}{d} \quad (45)$$
   where $\tilde{\boldsymbol{x}}_{l'}^S$ is the l'^{th} SIV
5. *Simulation of evidence sets.* The evidence-set, involves a combination of r SIVs. Therefore, the multinomial distribution can be used to calculate the probability of each combination, and hence each possible evidence-set. Evidence is of the form shown in Table **1**, recalling that z refers to the number of states the system's components can exist in. Evidence gathered by the i^{th} sensor is of the form $\tilde{\boldsymbol{k}}_i^S = \{k_i^{S(0)}, k_i^{S(1)}, k_i^{S(2)}, \dots, k_i^{S(z-1)}\}$ observed states out of r demands. For a set of r SIVs, the evidence, E, can be properly defined using Equation 46.
   $$E = \{\tilde{\boldsymbol{k}}_1^S, \tilde{\boldsymbol{k}}_2^S, \tilde{\boldsymbol{k}}_3^S, \dots, \tilde{\boldsymbol{k}}_i^S, \dots, \tilde{\boldsymbol{k}}_m^S, r\} \quad (46)$$
   where, $k_i^{S(x)} = \sum_{l'=1}^{r} \begin{cases} 1 \; if \; (x_i^S)_l = x \\ 0 \; otherwise \end{cases}$

Monte-Carlo simulation, based on the probabilities of each SIV from Equation 45, can simulate evidence sets.

$$\Pr(E_q|\varepsilon,\pi_0) \approx \frac{d_q}{d} \quad (47)$$

where, d_q is the number of simulations that imply the q^{th} evidence set, E_q, out of a total of d simulations

The steps for continuously operating systems are:

i. *Sampling unknowns of interest* – $\boldsymbol{\theta}$. Monte-Carlo simulation can be used to randomly draw joint samples of the unknowns of interest.

ii. *Simulating component failure times* – $\tilde{\boldsymbol{t}}$. A *PDF* and *CDF* are defined by each draw of the unknowns of interest, $\widehat{\boldsymbol{\theta}}$. Random sampling based on these functions simulates component failure times.

$$\tilde{\boldsymbol{t}}_l = \left\{(t_1)_l, (t_2)_l, (t_3)_l, \dots, \left(t_j\right)_l, \dots, (t_n)_l\right\} \quad (48)$$

where $\left(t_j\right)_l$ is the time to failure of the j^{th} component in the l^{th} simulation

iii. *Simulating failure detection times*. System logic allows the component failure times to determine systemic failure time.

$$\tilde{\boldsymbol{t}}^S_{l'} = \{(t^S_1)_{l'}, (t^S_2)_{l'}, (t^S_3)_{l'}, \dots, (t^S_i)_{l'}, \dots, (t^S_m)_{l'}\} \quad (49)$$

where, $(t^S_i)_{l'}$ is the time that failure is detected by the i^{th} sensor in the l'^{th} simulation

iv. *Simulating evidence sets - sorting failure detection times*. Sensor failure detection times should then be sorted to bins that correspond with their measurement accuracy. For example, if the sensor detects failures accurate to within a second, then the simulated evidence becomes the number of times that fall within second intervals.

g. **Simulation of posterior distributions.** The *information utility* of each simulated evidence data set requires is based on the posterior distribution of the unknowns of interest, as defined in Equation (50).

$$\pi_1^q\left(\boldsymbol{\theta}|E_q, \varepsilon\right) = \frac{L\left(E_q|\boldsymbol{\theta}, \varepsilon\right)\pi_0(\boldsymbol{\theta})}{\int_{\forall\boldsymbol{\theta}} L\left(E_q|\boldsymbol{\theta}', \varepsilon\right)\pi_0(\boldsymbol{\theta}')d\boldsymbol{\theta}'} \quad (50)$$

h. **Expected Information Utility.** The *information utility* function in Equation (41) along with the posterior distribution of each evidence set in Equation (50) and it probability of occurring in Equation (47) can then be substituted into Equation (37) to derive the expected *information utility* of the experimental framework, ε (*i.e.*, the sensor placement).

4 Examples of Application

The following examples illustrate the practical application of sensor placement optimization to maximize information utility only. They can easily be incorporated into multi-objective optimization methodologies that include other metrics such as cost.

4.1 Multiples Instances of Identical Components

Systems often involve multiple instances of the same component, particularly when used in parallel or other redundancy frameworks. In such cases, multiple component reliability characteristics will be described by the same set of parameters. For example, multiple instances of identical components in on-demand systems will see components' failure probabilities described be the same parameter - p. To accommodate this, analysis focuses on *component types* – not each component. For on-demand systems where the unknowns of interest (UOIs) are the failure probability parameters, there will be n UOIs and n' components. This approach is illustrated in the example below [7].

Consider the four-component binary state on-demand system illustrated in Figure 4 (*i.e.*,$n' = 4$) with two identical components. The UOIs are the component type failure probabilities p_1, p_2 and p_3 (*i.e.* $n = 3$). Sensors need to be placed within the system,

which incurs costs. One sensor is constrained to be placed at the system or 'top-event' level, with two possible sensor locations denoted 'Possible Sensor #2' and 'Possible Sensor #3'.

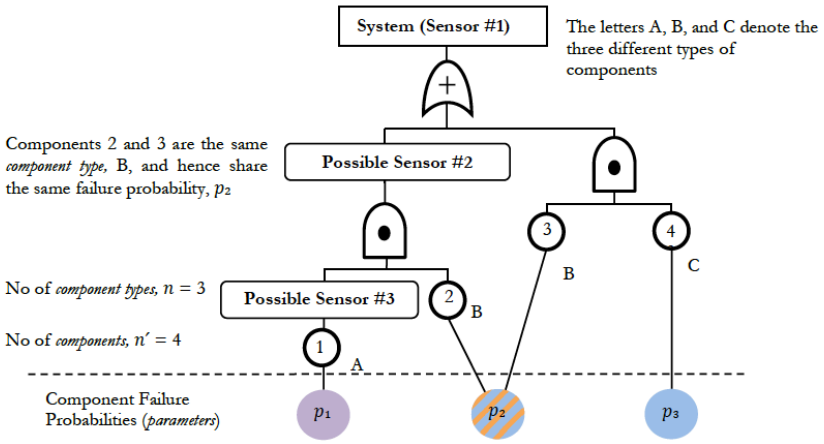

Figure 4: Four (4) Component (3-component type) Binary State On-demand System

A prior uniform distribution is assumed for each unknown of interest (UOI).

$$\pi_0(\boldsymbol{\theta}) = \pi_0(p_1, p_2, p_3) = \prod_{i=1}^{3} \pi_0(p_i)$$
$$\text{where } \pi_0(p_i) = \begin{cases} 1 \; if \; 0 \le p_i \le 1 \\ 0 \; otherwise \end{cases} \quad (51)$$

The information utility used to assess the value of various sensor placements is selected to be the inverse of the sum of the posterior variance of all parameters (from Table 2):

$$U^I(E, \varepsilon|\pi_0) = U^I(\pi_1|\pi_0) = \frac{1}{\sum_{j=1}^{n} \sigma^2_{\theta_j}} \text{ where } \boldsymbol{\theta} \sim \pi_1(\boldsymbol{\theta}|E, \varepsilon) \quad (52)$$

The possible sensor arrangements are:

$\varepsilon_1 = \{1\}$ … (systemic level failure detection only) (53)
$\varepsilon_2 = \{1,2\}$ … (systemic level and sensor #2 failure detection) (54)
$\varepsilon_3 = \{1,3\}$ … (systemic level and sensor #3 failure detection) (55)
$\varepsilon_4 = \{1,2,3\}$ … (systemic level, sensor #2 and #3 failure detection) (56)

The likelihood function for this on demand system is a modification of Equation (13) to incorporate multiple instances of the same component.

$$L(E|\boldsymbol{\theta}) \propto \sum_{\forall \tilde{v}_a \in \mathbf{v}_E} \left[\prod_{l=1}^{z^{n'}} \frac{1}{(v_l)_a!} \left(\prod_{b=1}^{n'} p_{j_b}^{(x_b)_l} \right)^{(v_l)_a} \right] \quad (57)$$

where j_b is the component type number of the b^{th} component

The reliability engineer selects a simulated test regime that is based on $r = 5$ demands. The structure functions for each possible sensor location are:

Sensor #1: $\phi_1(x_1, x_2, x_3, x_4) = \phi_2(x_1, x_2) + x_3 x_4 - x_3 x_4 \phi_2(x_1, x_2)$ (58)
Sensor #2: $\phi_2(x_1, x_2) = x_2 \phi_3(x_1)$ (59)
Sensor #3: $\phi_3(x_1) = x_1$ (60)

where x_1, x_2, x_3 and x_4 are the state variables of all four components

Monte-Carlo simulation is used to sample the set $\{p_1, p_2, p_3\}$ a large number of times based on the prior distribution in Equation (51). Each draw defines the state probabilities of each component, allowing component state vectors, $\tilde{\boldsymbol{x}}$, to be randomly drawn again. This process yields the following component state vectors with identical probabilities:

Table 3: All Possible Component State Vectors with Probabilities of Occurrence

$\Pr(\tilde{\boldsymbol{x}})$:		$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{12}$
$\tilde{\boldsymbol{x}}$	Component 1 (x_1):	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
	Component 2 (x_2):	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
	Component 3 (x_3):	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
	Component 4 (x_4):	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

Each component state vector generates a corresponding sensor information vector. Based on the data in Table 3, all possible SIVs along with their probabilities can be calculated. The SIVs in Table 4 are used for Monte-Carlo Simulation of evidence sets, E. In this example, 100,000 simulations are generated with 90 distinct evidence sets generated.

Table 4: All Possible Sensor Information Vectors with Probabilities of Occurrence

$\Pr(\tilde{\boldsymbol{x}}^S)$:		$\frac{3}{8}$	$\frac{5}{24}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{24}$
$\tilde{\mathrm{X}}^S$	Sensor 1 $(\boldsymbol{x}_1^S)$:	0	0	1	1	1
	Sensor 2 $(\boldsymbol{x}_2^S)$:	0	0	1	0	0
	Sensor 3 $(\boldsymbol{x}_3^S)$:	0	1	1	0	1

Consider the first sensor placement framework, ε_1, with sensors at all three possible locations. Each simulated evidence set along with Equation (57) can be substituted into Equation (50) to yield a posterior distribution. The utility function for each posterior distribution is given in Equation (52). The five most probable of these evidence sets are listed in Table 5, along with the utility of each. Equation (52) can then be used to generate the sensor placement expected utility:

$$\bar{U}^I(\varepsilon_1|\pi_0) = \sum_{\forall q} U_q^I \cdot \Pr\left(E_q|\varepsilon_1, \pi_0\right) = 23.746 \tag{61}$$

Table 5: Possible Evidence Sets with Probabilities of Occurrence (10 most Probable of 90 Evidence Sets Simulated)

Evidence Set Number, q	**Probability of evidence set occurring, $\Pr\left(E_q|\varepsilon, \pi_0\right)$**	**Evidence set, E_q (based on r = 5 demands)**			**Utility, U_q^I**
		Number of observed failures by Sensor #1	**Number of observed failures by Sensor #2**	**Number of observed failures by Sensor #3**	
1	0.065918	2	1	2	22.061
2	0.054932	2	2	3	21.745
3	0.054932	1	1	2	20.090
4	0.048828	2	1	3	23.576
5	0.047607	3	2	3	23.856
…					

The same process is used for all other sensor arrangements. The resultant expected utility for each is summarized in Table 6.

Table 6: Expected Information Utility for Each Sensor Arrangement

i	ε_i	**Expected Information Utility,** $\bar{U}^I(\boldsymbol{\varepsilon}_i\|\boldsymbol{\pi}_0)$
	none	11.964
1	{1}	14.838
2	{1,2}	17.558
3	{1,3}	20.645
4	{1,2,3}	23.746

It can be seen that using all three possible sensor locations yields the most information. However, if the total number of sensors that could be used is limited to two, than it is clearly most beneficial for locations 1 and 3 to be used, as opposed to locations 1 and 2. This is because sensors at locations 1 and 3 can still yield significant information about component 2 in addition to yielding more detailed information about component 1.

As another example, consider the system examined in the previous example, however suppose it is now continuous and time based. Only one sensor can be installed: sensor #1 location (for \$1,000) or sensor #2 location (for \$100). Component A has a constant failure rate, λ_1. Components B and C have constant failure rate, λ_2. Component D's time to failure is normal distributed with mean μ_3 and standard deviation σ_3. The set of unknowns of interest is therefore:

$$\boldsymbol{\theta} = \{\lambda_1, \lambda_2, \mu_3, \sigma_3\} \tag{62}$$

Each unknown of interest has a uniform prior distribution with the following limits:

$$\lambda_1 \text{ and } \lambda_2 \sim U(0,10) \tag{63}$$

$$\mu_3 \sim U(10,15) \tag{64}$$

$$\sigma_3 \sim U(0,5) \tag{65}$$

A single test has already been conducted with the following evidence set:

$$E = \begin{cases} t_1^S = 0.8 & \dots \quad \Delta t_1 = 0.1 \\ t_2^S = 0.7 & \dots \quad \Delta t_2 = 0.2 \\ t_3^S = 0.5 & \dots \quad \Delta t_3 = 0.1 \end{cases} \tag{66}$$

The likelihood function is developed in Equation (28).

$$L\left(\hat{\tilde{\boldsymbol{t}}}^S|\boldsymbol{\theta}, \Delta t_1, \dots, \Delta t_m\right) = L\left(\{\hat{t}_1^S, \dots, \hat{t}_m^S\}|\{\tilde{\boldsymbol{\theta}}_1, \dots, \tilde{\boldsymbol{\theta}}_n\}, \{\Delta t_1, \dots, \Delta t_m\}\right) \tag{28}$$

$$= \prod_{i=1}^{m} [F_i^S(\hat{t}_i^S + \Delta t_i|\boldsymbol{\theta}_i^{\subset S}, \tilde{\boldsymbol{t}}_i^{\subset S}) - F_i^S(\hat{t}_i^S|\boldsymbol{\theta}_i^{\subset S}, \tilde{\boldsymbol{t}}_i^{\subset S})]$$

The resulting state of knowledge yields the following marginal distributions for the unknowns of interest. $\pi_1\left(\boldsymbol{\theta}|\hat{\tilde{\boldsymbol{t}}}^S, \Delta t_1, \dots, \Delta t_m\right)$

The information utility used to assess the value of various sensor placements is selected to be the inverse of the sum of the posterior variance of all parameters:

$$U^I(E, \varepsilon|\pi_0) = U^I(\pi_1|\pi_0) = \frac{1}{\sum_{j=1}^{n} \boldsymbol{\sigma}_{\theta_j}^2} \text{ where } \boldsymbol{\theta} \sim \pi_1(\boldsymbol{\theta}|E, \varepsilon) \tag{67}$$

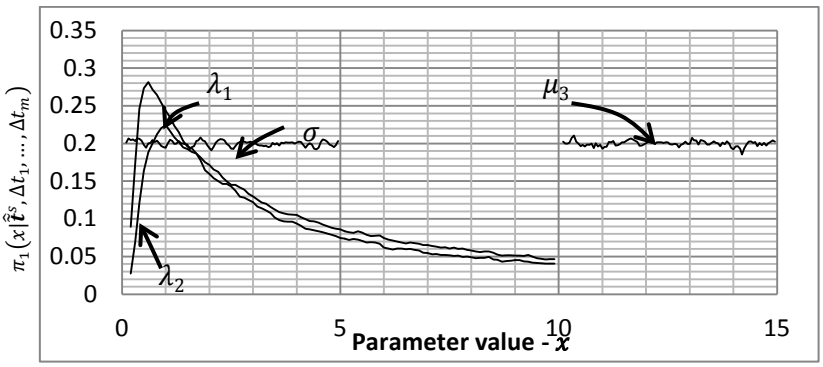

Figure 5: Marginal Distributions of the Unknowns of Interest generated by the Current State of Knowledge (1 000 000 Simulations, Slice Sampling Algorithm of interval 5)

The same state of knowledge yields the time to failure distributions illustrated in Figure 6.

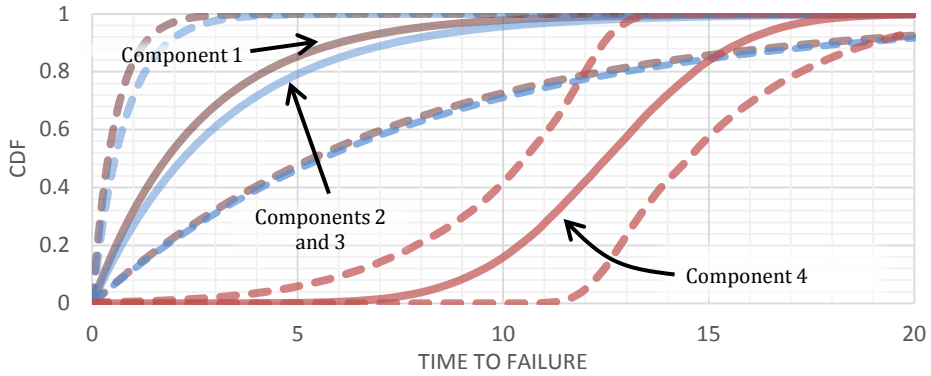

Figure 6: Time to Failure Distributions for Components 1, 2, 3 and 4 based on the State of Knowledge represented in Figure 5 (80 per cent Confidence Bounds illustrated)

The possible sensor arrangements are:

$\varepsilon_1 = \{1\}$ … (systemic level failure detection only) (69)

$\varepsilon_2 = \{2\}$ … (sensor #2 failure detection only) (70)

The structure functions remain unchanged from the previous example where failure probabilities replace state variables. Monte-Carlo simulation is used to sample the set $\{\lambda_1, \lambda_2, \mu_3, \sigma_3\}$ based on the state of knowledge illustrated in Figure 5. Each sample defines the PDF and CDF of each component, allowing component failure times (and sensor failure detection times, t_i^S) to be randomly drawn again. This process yields simulated sensor detection times for each sensor placement arrangement as listed in Table 7.

Each simulated evidence set along with Equation (28) can be substituted into Equation (50) to yield a posterior distribution. The utility function for each posterior distribution is given in Equation (67). Equation (36) can then be used to generate the expected utility of each sensor placement arrangement, and the data is summarized in Table 8.

Table 7: Simulated Failure Detection Times Based on Current State of Knowledge (five most probable evidence sets displayed based on 100,000 simulations)

$\boldsymbol{\varepsilon_1}$ = {1} … Sensor # 1		$\boldsymbol{\varepsilon_2}$ = {2} … Sensor # 2	
Failure Detection Time	**Probability**	**Failure Detection Time**	**Probability**
$\mathbf{0 \le t_1^S < 0.5}$	0.4373	$\mathbf{0 \le t_2^S < 0.5}$	0.4369
$\mathbf{0.5 \le t_1^S < 1.0}$	0.2517	$\mathbf{0.5 \le t_2^S < 1.0}$	0.2517
$\mathbf{1.0 \le t_1^S < 1.5}$	0.1172	$\mathbf{1.0 \le t_2^S < 1.5}$	0.1172
$\mathbf{1.5 \le t_1^S < 2.0}$	0.0618	$\mathbf{1.5 \le t_2^S < 2.0}$	0.0618
…	**…**	…	**…**

Table 8: Expected Information Utility for Each Sensor Arrangement

$\boldsymbol{i}$	$\boldsymbol{\varepsilon_i}$	**Expected Information Utility,** $\boldsymbol{\bar{U}^I(\varepsilon_1 \mid \pi_0)}$
1	**{1}**	1.141
2	**{2}**	1.128

It can be seen that the expected utility for each sensor is very similar - component 4 will almost certainly fail after all of the other components (as illustrated in Figure 6). Therefore, sensors at locations 1 and 2 will certainly detect failure. The sensor at location 1 has lower utility as it is predicated on the nature of component 4, and in effect is improving the understanding of the component 4 failure characteristics, as it will suggest that component 4 hasn't failed when sensor #2 would otherwise detect failure. The sensor at location 2 yields information directly onto the remaining 3 components only.

Notwithstanding the slight difference in information utility, the cost of installing a sensor at the second possible location is significantly less than that for the first possible location. Therefore, it is probably most valuable (in both an information utility and cost/benefit perspective) to install the sensor at the second possible location.

5 Conclusion

Engineering systems are becoming increasingly complex with ever increasing functionality. Our dependence on these complex systems also sees the 'value' of the functions these systems provide increasing, which by extension increases the 'cost' of failure. Sensors are becoming increasingly affordable and useful, but they are not free and require careful consideration in terms of where they are located within systems and for what aims. Sensor data gathered simultaneously from the same system are 'overlapping' in nature. Overlapping sensor data is contextualized by, and itself contextualizes the data of all other sensors.

This paper presented an overview of how the analysis of overlapping sensor data is implemented to infer information regarding component level reliability parameters. The improved understanding of these parameters then informs system level reliability performance. This paper also outlined how prior information can be used to develop information 'utility' associated with a particular arrangement of sensors within a system. Evidence data sets can be simulated with the inferred information quantified to inform the utility value. By calculating the information utility for each sensor arrangement, the optimum sensor configuration can be identified and either directly implemented, or used

within a larger multi-objective optimization process. Finally, two examples were presented to demonstrate the application of the reviewed methodologies.

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