Belief Reliability for Uncertain Random Systems

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A short introduction
School of Reliability and Systems Engineering, BUAA

Education
- 90 undergraduates every year
- 150 graduate students every year
- 40 Phd. candidates every year
- 120 faculty members

Research
- More than 100 scientific research and hi-tech projects every year

Consultation
- As a national think tank, provides policy advice to the government on reliability technology and engineering

Engineering
- Provide a large number of technical services for industry

- National Key Laboratory for Reliability and Environmental Engineering
- Department of Systems Engineering of Engineering Technology
- Department of System Safety and Reliability Engineering
- Center for Product Environment Engineering
- Center for Components Quality Engineering
- Center for Software Dependability Engineering
Double Helix Structure of Reliability Science

**Abstract Objects**
- Cyber Physics Social System
- Cyber Physics System
- Network
- Hardware+Software
- Hardware & Software

**Methodology**
- Failure/Fault Prophylaxis
- Failure/Fault Diagnostics
- Failure/Fault Prognostics
- Failure/Fault Cybernetics

**Failurology**
- Recognize Failure Rules & Identify Failure Behaviors
Outline

Research Background

Requirements Analysis

Theoretical Framework

Conclusion & Future
Reliability

**Definition:** Reliability refers to the ability of a component or a system to perform its required functions under stated operating conditions for a specified period of time.

**Four basic problems:** Reliability metric, analysis, design and verification
Uncertainty

**Classification:** Aleatory uncertainty & Epistemic uncertainty

**Aleatory uncertainty**
Inherent randomness of the physical world and cannot be eliminated. This kind of uncertainty is also called random uncertainty.

**Epistemic uncertainty**
Uncertainty due to lack of knowledge. It can be reduced through scientific and engineering practices.

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Source of epistemic uncertainty

Example - Software
Probability theory

Probability measure

Probability Theory (Kolmogorov, 1933)

Axiom 1. Normality Axiom: For the universal set $\Omega$, $\Pr\{\Omega\} = 1$.

Axiom 2. Nonnegativity Axiom: For any event $A$, $\Pr\{A\} \geq 0$.

Axiom 3. Additivity Axiom: For every countable sequence of mutually disjoint events $\{A_i\}$, we have

$$\Pr\left\{\bigcup_{k=1}^{\infty} A_i\right\} = \sum_{k=1}^{\infty} \Pr\{A_i\}.$$ 

Product Probability Theorem: For any probability space $(\Omega_k, \mathcal{A}_k, \Pr_k), k = 1, 2, ...$,

$$\Pr\left\{\prod_{k=1}^{\infty} A_k\right\} = \prod_{k=1}^{\infty} \Pr_k\{A_k\}.$$ 

where $A_k$ are arbitrarily chosen events from $\mathcal{A}_k, k = 1, 2, ...$
**Bernoulli’s Law of Large Numbers** (Bernoulli, 1713)

Let $\mu$ be the occurrence times of event $A$ in $n$ independent experiments. If the probability that event $A$ occurs in each test is $p$, then for any positive number $\varepsilon$:

$$\lim_{n \to \infty} \Pr \left\{ \left| \frac{\mu}{n} - p \right| < \varepsilon \right\} = 1.$$
Classical probabilistic reliability metric

At the very beginning…

• Probability theory is used to represent uncertainty
• In World War II, German rocket scientist Robert Lusser advocated the probability product rule

System reliability is the product of the reliability of each subsystem.

$$R_S = R_1 \cdot R_2 \cdot \ldots \cdot R_n$$

R. Lusser (1899-1969)
Classical probabilistic reliability metric

Black box method: Probabilistic metric based on failure data

- Features: The reliability is calculated using statistical methods. This method doesn’t separate aleatory and epistemic uncertainty.
- Shortage: We must collect enough failure time data. It is hard to indicate how to improve reliability.

Classical probabilistic reliability metric

White box method: Probabilistic metric based on physics of failure

- Physics-of-failure models (PoF models)
  
  A PoF model is a mathematical model that quantifies the relationship between failure time or performance and product’s features, such as material, structure, load, stress, etc. It is developed for one specific failure mechanism based on physics and chemistry theories.

- A simple example – Archard’s model (wear life model)

\[ N = \frac{h_sHA}{\mu W_aL_m} \]

- Failure time: \( N \): Wearing times
- Material: \( H \): Hardness
- Structure: \( \mu \): Dynamic friction coefficient
- Load: \( A \): Contact area of two wear surfaces
- Threshold: \( W_a \): Contact pressure
- \( h_s \): The max acceptable wear volume
Classical probabilistic reliability metric

White box method: Probabilistic metric based on physics of failure

- **Features:** The failure is described by a deterministic model. The uncertainty only comes from the variability of model parameters. This method is able to measure reliability when there’s few data. The results can guide design improvements.

- **Shortage:** The method may overestimate the reliability by ignoring epistemic uncertainty.

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Classical probabilistic reliability metric

White box method: Source of epistemic uncertainty

- Functional principle
- Failure mechanism
- PoF model
  \[ TF = f(x_1, x_2, \ldots) \]
- Variability of parameters

- Lack of knowledge about the product function and failure mechanism
- Lack of knowledge about the product working conditions

Model uncertainty

Parameter uncertainty

Reliability metric considering epistemic uncertainty

Reliability metric considering EU

Imprecise probabilistic reliability metric
- Bayes theory — Bayesian reliability
- Evidence theory — Evidence reliability
- Interval analysis — Interval reliability

Fuzzy reliability metric
- Fuzzy theory — Fuzzy reliability
Reliability metric considering EU

Imprecise probabilistic reliability metric
- Bayes theory — Bayesian reliability
- Evidence theory — Evidence reliability
- Interval analysis — Interval reliability

Posbist reliability metric
- Possibility theory — Posbist reliability
Reliability metric considering EU

Imprecise probabilistic metric: Bayesian reliability

- Theoretical basis – Bayes theorem

\[ p(\theta|y) = \frac{f(y|\theta)p(\theta)}{m(y)} \]

- How to consider EU?

Our knowledge on the failure process is reflected in the different forms of prior distribution.

Reliability metric considering EU

Imprecise probabilistic metric: Bayesian reliability

- Method to obtain reliability

\[ R(t) = \int_{t}^{\infty} f_T(\xi | \theta) d\xi \]

- \( f_T(t | \theta) \): pdf of failure time \( T \)
- \( p(\theta) \): prior distribution of parameter \( \theta \)
- \( t \): some failure time data

- \( p(\theta | t) \): posterior distribution of \( \theta \)
- \( R_m(t) \): Use the median reliability \( R_m(t) \) as the reliability index

Reliability metric considering EU

Imprecise probabilistic metric: Evidence reliability

- Theoretical basis – Evidence theory
  - Proposed by A. Dempster and G. Shafer and refined by Shafer.
  - Use evidence to calculate **Belief** and **Plausibility** $\rightarrow$ **Probability interval**

  \[
  \text{Bel}(A) \leq P(A) \leq \text{Pl}(A)
  \]

Fig. Belief and Plausibility

- How to consider EU?
  Experts may set **basic probability assignment (BPA)** to different values of the model parameters based on experience or similar product information, reflecting the belief degree of the corresponding values.

Reliability metric considering EU

Imprecise probabilistic metric: Evidence reliability

- Method to obtain reliability

  Construct a performance model
  \[ y = g(x_1, x_2, \ldots) \]

  Identify the failure region \( \{ y \leq y_{th} \} \)
  Event \( A \): system is working

  Define the frame of discernment and assign BPAs to possible values of parameters

  Calculate probability interval \([Bel(A), Pl(A)]\)

  For example, \( y \) represents output voltage and \( y = g(x_1, x_2) = x_1^2x_2/20 \)

  Let \( y_{th} = 1V \), then \( A = \{y \geq 1V\} \)
  denotes working state

  \[ \Theta = \{ [2, 4] \times [2, 4] \} \]

  \[
  \begin{array}{ccc}
  x_1 & \text{Intervals} & \text{BPA} \\
  [2.0, 2.5] & 0.0478 & [2.0, 2.5] \\
  [2.5, 3.0] & 0.4522 & [2.5, 3.0] \\
  [3.0, 3.5] & 0.4522 & [3.0, 3.5] \\
  [3.5, 4.0] & 0.0478 & [3.5, 4.0] \\
  \end{array}
  \]

  \[ 0.5 \leq P(A) \leq 0.976 \]

Reliability metric considering EU

Imprecise probabilistic metric: Interval reliability

- Theoretical basis – Interval analysis
  - Proposed by Ramon E. Moore.
  - Calculate the interval of model output based on intervals of input parameters

\[
\begin{align*}
\text{Input parameter} & \quad x_L \leq x \leq x_U \\
\text{Model} & \quad y = f(x) \\
\text{Model output} & \quad y_L \leq y \leq y_U
\end{align*}
\]

- How to consider EU?
  - The expert may give the upper and lower bounds of the model parameters based on experience or similar product information. Parameters can take any values within the given interval. **The width of the interval reflects the degree of epistemic uncertainty.**

Reliability metric considering EU

Imprecise probabilistic metric: Interval reliability

Method to obtain reliability

Construct a performance model
\[ y = f(x_1, x_2, \ldots) \]

The upper and lower bounds of distributions are given by experts
\[ [\mu_{iL}, \mu_{iU}], [\sigma_{iL}, \sigma_{iU}], \ldots \]

Construct a p-box of \( y \)

Algorithm:
- Cartesian product method\(^1\)
- Optimization method\(^2\)

\[ p = P(y \leq y_{th}) = F_Y(y_{th}) \]

Then we have \([p_L, p_U]\) and \([R_L, R_U]\)

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Reliability metric considering EU

Shortages of Imprecise probabilistic metric

- Interval extension problem

Example

Consider a series system composed of 30 components. Suppose that the reliability interval for each component is [0.9,1]. Then, the system’s reliability metric will be $[0.9^{30}, 1^{30}] = [0.04, 1]$, which is obviously too wide to provide any valuable information in practical applications.

- Disconnection between macro and micro

  The metrics doesn’t show the relationship between reliability and product design parameters. Therefore, their abilities to guide the improvement of products are very limited.
Reliability metric considering EU

Imprecise probabilistic reliability metric
- Bayes theory — Bayesian reliability
- Evidence theory — Evidence reliability
- Interval analysis — Interval reliability
- Fuzzy set theory — Fuzzy interval reliability

Posbist reliability metric
- Possibility theory — Posbist reliability
In possibility theory, the possibility measure $\Pi$ satisfies three axioms:

**Axiom 1.** For the empty set $\emptyset$, $\Pi(\emptyset) = 0$,

**Axiom 2.** For the universal set $\Gamma$, $\Pi(\Gamma) = 1$,

**Axiom 3.** For any events $A_1$ and $A_2$ in the universal set $\Gamma$, there is

$$\Pi(A_1 \cup A_2) = \max(\Pi(A_1), \Pi(A_2))$$

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Reliability metric considering EU

Fuzzy reliability metric

Mathematical measure
- PRObability measure
- POSsibility measure

System state
- BIInary STate
- FUuzzy STate

- Probist reliability
- Profust reliability
- Posbist reliability
- Posfust reliability

Reliability metric considering EU

Posbist reliability metric

- Basic assumption
  - **Possibility assumption**
    System failure behavior can be characterized under possibility
  - **Binary-state assumption**
    The system demonstrates only two crisp states: functioning or failed

- Definition

  **Posbist Reliability (Cai, 1991)**

  Suppose the system failure time $T$ is a fuzzy variable. Then the posbist reliability at time $t$ is defined as the possibility measure that $T$ is greater than $t$:

  $$R(t) = \Pi(T \geq t)$$

- How to consider EU?

  The failure time is modeled as a fuzzy variable, and the possibility distribution of failure time describes the epistemic uncertainty.

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Reliability metric considering EU

Shortages of posbist reliability metric

- Non-duality

Example

Consider two exclusive events: $\Lambda_1 = \{\text{The system is working}\}$, $\Lambda_2 = \{\text{The system fails}\}$. Obviously, the universal set $\Gamma = \{\Lambda_1, \Lambda_2\}$. Then, we have the posbist reliability and posbist unreliability to be $R_{\text{pos}} = \Pi (\Lambda_1)$ and $\overline{R_{\text{pos}}} = \Pi (\Lambda_2)$.

According to Axiom 2 and Axiom 3, it can be proved that:

$$\Pi (\Gamma) = \Pi (\Lambda_1 \cup \Lambda_2) = \max (\Pi (\Lambda_1), \Pi (\Lambda_2)) = \max (R_{\text{pos}}, \overline{R_{\text{pos}}}) = 1$$

Therefore, if $R_{\text{pos}} = 0.8$, then $\overline{R_{\text{pos}}} = 1$, and if $\overline{R_{\text{pos}}} = 0.8$, then $R_{\text{pos}} = 1$. This result is counterintuitive.

Outline

Research Background

Requirements Analysis

Theoretical Framework

Conclusion & Future
Requirements for reliability metric

A reliability metric must satisfy the normality principle, i.e., the sum of measurement of all states should be equal to 1. Specially, reliability plus unreliability must be 1.

This is mathematically consistent, also logically consistent. It can avoid the bug of fuzzy reliability.

Requirements for reliability metric

A reliability metric should be able to be used not only for the reliability evaluation of components and simple systems, but also for that of complex systems. When it is used for reliability calculation of the system, it cannot decrease as quickly as interval-based method, i.e., it should be able to compensate the conservatism in the component level.

Requirements for reliability metric

A reliability metric must enable multiscale analysis. The bridge between reliability metric and product or system design elements can be established through multiscale analysis. This can provide more feedback on improving product or system reliability and avoids the embarrassment in statistical methods because statistical methods only give the results but don't know why.

Requirements for reliability metric

A reliability metric should be able to support the uncertain information fusion. The reliability information is available early in the design phase of a product. At this time, the degree of epistemic uncertainty is very high. As the design process advances, epistemic uncertainty will gradually decrease with a relative increase of aleatory uncertainty. The reliability metric must be able to integrate these different information.

Requirements for reliability metric

- **Theoretical Completeness**
  - **R1:** Normality
  - **R2:** Slow decrease

- **Belief reliability theory**

- **Engineering Practicability**
  - **R3:** Multiscale analysis
  - **R4:** Information fusion
Outline

- Research Background
- Requirements Analysis
- Theoretical Framework
- Conclusion & Future
Preliminary about math theory
Belief reliability metric

Theoretical basis: Uncertainty theory

Uncertainty Theory (Liu, 2007)

In uncertainty theory, the uncertainty measure $\mathcal{M}$ satisfies the following 4 axioms:

**Axiom 1. Normality axiom:** For the universal set $\Gamma$, $\mathcal{M}\{\Gamma\} = 1$.

**Axiom 2. Duality axiom:** For any event $\Lambda$, $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$.

**Axiom 3. Subadditivity axiom:** For every countable sequence of events $\Lambda_1, \Lambda_2, ...$,

$$
\mathcal{M}\left(\bigcup_{k=1}^{\infty} \Lambda_i\right) \leq \sum_{k=1}^{\infty} \mathcal{M}\{\Lambda_i\}.
$$

**Axiom 4. Product axiom:** For any uncertainty space $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k), k = 1, 2, ...$,

$$
\mathcal{M}\left(\prod_{k=1}^{\infty} \Lambda_k\right) = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}.
$$

where $\Lambda_k$ are arbitrarily chosen events from $\mathcal{L}_k, k = 1, 2, ...$

General theoretical basis

Chance theory

Chance theory (Liu, 2013)

Chance theory defines chance measure $Ch$, which can be regarded as a mixture of probability measure and uncertainty measure.

Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ be a chance space, and $\Theta \in \mathcal{L} \times \mathcal{A}$ is an event over this space. Then, the chance measure of $\Theta$ is defined to be:

$$Ch\{\Theta\} = \int_0^1 Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Omega\} \geq x\} dx$$

Chance theory

**Chance measure (Liu, 2013)**

Let \((\mathcal{G}, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)\) be a chance space, and \(\Theta \in \mathcal{L} \times \mathcal{A}\) is an event over this space. Then, the chance measure of \(\Theta\) is defined to be:

\[
\text{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \mathcal{G} | (\gamma, \omega) \in \Omega\} \geq x\} \, dx
\]

**Theorem**

Let \((\mathcal{G}, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)\) be a chance space, then for any \(\Lambda \in \mathcal{L}\) and \(A \in \mathcal{A}\):

\[
\text{Ch}\{\Lambda \times A\} = \mathcal{M}\{\Lambda\} \times \Pr\{A\}.
\]

Especially we have \(\text{Ch}\{\emptyset\} = 0\), \(\text{Ch}\{\mathcal{G} \times \Omega\} = 1\).

**Definition (Uncertain random variable)**

An uncertain random variable is a function $\xi$ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers such that $\{\xi \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set $B$ of real numbers.

- $\xi$ can degenerate to a random variable if $\xi(\gamma, \omega)$ does not vary with $\gamma$.
- $\xi$ can degenerate to an uncertain variable if $\xi(\gamma, \omega)$ does not vary with $\omega$. 
Definition (Chance distribution)

Let $\xi$ be an uncertain random variable, then its chance distribution is defined by

$$\Phi(x) = \text{Ch}\{\xi \leq x\}$$

for any $x \in \mathbb{R}$. It can also degenerate to either probability or uncertainty distribution.

Definition (Expected value and variance)

Let $\xi$ be an uncertain random variable, then its expected value is defined by

$$E[\xi] = \int_{0}^{+\infty} \text{Ch}\{\xi \geq x\}dx - \int_{-\infty}^{0} \text{Ch}\{\xi \leq x\}dx,$$

provided that at least one of the two integrals is finite. Suppose $\xi$ has an finite expected value $e$, the variance of $\xi$ is defined as

$$V[\xi] = E[(\xi - e)^2].$$
Concepts and definitions of belief reliability
**Uncertain random systems**

**Definition:** The system composed of uncertain and random components

- **Uncertain components:** Components affected by severe epistemic uncertainty. Their reliability can be described by uncertainty theory.
- **Random components:** Components mainly affected by aleatory uncertainty with sufficient failure data. Their reliability should be modeled by probability theory.
Belief reliability analysis of cloud data center
## Belief reliability analysis of cloud data center

### Parameter Setting - Certain Parameters

#### Parameters Related to the Design of CDC

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Setting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Function of Protocol and Routing Rules $F_{PR}$</td>
<td></td>
</tr>
<tr>
<td>Number of Clusters and Sub-Clusters $K, J$</td>
<td></td>
</tr>
<tr>
<td>Number of Subtasks $Y_{kA1}, Y_{kA1}, Y_{kD}$</td>
<td></td>
</tr>
<tr>
<td>Number of VMs for Each Physical Machine $N_{VM}$</td>
<td>According to the construction</td>
</tr>
<tr>
<td>Number of Active Redundancy for Each Node $N_{R}$</td>
<td></td>
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<tr>
<td>Number of Hot Standby for Each Node $N_{HS}$</td>
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</tbody>
</table>
# Belief reliability analysis of cloud data center

**Parameter Setting - Uncertain Parameters**

Parameters Related to the Operation and Maintenance of CDC

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Setting</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Working Probability $pr$</td>
<td>Evaluated through monitoring data</td>
<td>Aleatory Uncertainty</td>
</tr>
<tr>
<td>Distribution Parameter of Processing Time $\lambda_s$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buffer Size $Q$</td>
<td>Estimated by experts</td>
<td>Epistemic Uncertainty</td>
</tr>
<tr>
<td>Recovery Time $\Delta t_r$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Distribution Parameter of Arrival Time $\lambda_{ak}$</td>
<td>Evaluated through monitoring data</td>
<td>Aleatory Uncertainty</td>
</tr>
</tbody>
</table>
Definition and connotation of BR

**Definition (Belief reliability)**

Let a system state variable $\xi$ be an uncertain random variable, and $\Xi$ be the feasible domain of the system state. Then the belief reliability is defined as the chance that the system state is within the feasible domain, i.e.,

$$R_B = Ch\{\xi \in \Xi\}$$

**Remark 1: $\xi$ and $\Xi$**

- The state variable $\xi$ describe the system behavior (function or failure behavior), and the feasible domain $\Xi$ is a reflection of failure criteria.
- $\xi$ and $\Xi$ can be relevant to time $t$, thus the belief reliability is a function of $t$, called belief reliability function $R_B(t)$.

**Remark 2: Two special cases**

- If the system is mainly affected by AU, $\xi$ will degenerate to a random variable, and the belief reliability becomes $R_B^{(P)} = Pr\{\xi \in \Xi\}$
- If the system is mainly affected by EU, $\xi$ will degenerate to an uncertain variable, and the belief reliability becomes $R_B^{(U)} = M\{\xi \in \Xi\}$

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**Definition and connotation of BR**

**Connotation 1:** The state variable represents failure time

The system state variable can represent system failure time $T$ which describes system failure behaviors. Therefore, the system belief reliability at $t$ can be obtained by letting the feasible domain of $T$ to be $\Xi = [t, +\infty)$, i.e.,

$$R_B(t) = \text{Ch}\{T > t\}.$$

**Example** (Belief reliability based on failure time)

The system state variable can represent system failure time $T$ which describes system failure behaviors. Therefore, the system belief reliability at $t$ can be obtained by letting the feasible domain of $T$ to be $\Xi = [t, +\infty)$, i.e.,

$$R_B(t) = \text{Ch}\{T > t\}.$$

**Two Special cases**

- If the system is mainly affected by AU, the failure time will be modeled as a random variable $T^{(P)}$, and we have $R_B(t) = R_B^{(P)}(t) = \Pr\{T^{(P)} > t\}$.

- If the system is mainly affected by EU, the failure time will be modeled as an uncertain variable $T^{(U)}$, and we have $R_B(t) = R_B^{(U)}(t) = \mathcal{M}\{T^{(U)} > t\}$.

**Definition and connotation of BR**

**Connotation 2:** The state variable represents performance margin

<table>
<thead>
<tr>
<th>Example (Belief reliability based on performance margin)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The system state variable can represent the performance margin ( m ) which describes system function behaviors. Let the feasible domain of ( m ) be ( \Xi = (0, +\infty) ), and the system belief reliability can be written as:</td>
</tr>
<tr>
<td>( R_B = \text{Ch}{m &gt; 0} ).</td>
</tr>
<tr>
<td>If we consider the degradation process of ( m ), then the belief reliability function is</td>
</tr>
<tr>
<td>( R_B(t) = \text{Ch}{m(t) &gt; 0} ).</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
m(t) & \quad \Rightarrow \quad T = t_0 = \inf\{t \geq 0 | m(t) = 0\} \quad \Rightarrow \quad R_B(t) = \text{Ch}\{m(t) > 0\} \\
\text{Uncertain random process} & \quad \quad \quad \text{Failure time is just the first hitting time of uncertain random process} \quad \quad \quad \text{Ch}\{t_0 > t\} \quad \quad \quad \text{Ch}\{T > t\}
\end{align*}
\]

**Definition and connotation of BR**

**Connotation 3:** The state variable represents function level

**Example (Belief reliability based on function level)**

The system state variable can represent the function level $G$ which describes both system function and failure behaviors, then it can measure the reliability of multi-state systems. Assume the system has $k$ different function levels with a lowest acceptable level of $G = s$. Let the feasible domain to be $\Xi = \{s, s + 1, \ldots, k\}$, then the system belief reliability is

$$R_B = Ch\{G \in \{s, s + 1, \ldots, k\}\}.$$

**Special case**

If the system has only two function levels, namely, complete failure with $G = 0$ and perfectly function with $G = 1$, then the belief reliability will be

$$R_B = Ch\{G = 1\}.$$ 

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Framework

- Belief reliability
  - Failure time
  - Performance margin
  - Function level

- Probability theory
- Uncertainty theory
- Chance theory

- Big data
- Spare data
- Model uncertainty
- Parameter uncertainty
- Boolean system
- Multi-state system

- Probability theory
- Uncertainty theory
Belief reliability indexes

\[ \Phi(x) \]
BR distribution

\[ R_B(t) \]
BR function

\[ T(\alpha) \]
Belief reliable life

\[ \text{MTTF} = \int_0^\infty R_B(t) \, dt \]
Mean time to failure

\[ \text{MTTF} = \int_0^1 T(\alpha) \, d\alpha \]

\[ \text{BLV} = \int_0^1 (T(\alpha) - \text{MTTF})^2 \, d\alpha \]
Belief life variance

## Some belief reliability indexes

### Belief reliability distribution

**Definition (Belief reliability distribution)**

Assume that a system state variable $\xi$ is an uncertain random variable, then the chance distribution of $\xi$, i.e.,

$$\Phi(x) = \text{Ch}\{\xi \leq x\}$$

is defined as the belief reliability distribution.

If the state variable represents the system failure time, the BRD will be the chance distribution of $T$, denoted as $\Phi(t)$. It can degenerate to either probability or uncertainty distribution.

If the state variable represents the system performance margin, the RBD will be the chance distribution of $m$, denoted as $\Phi(x)$. It can degenerate to either probability or uncertainty distribution.
Belief reliable life

Definition (Belief reliable life)

Assume the system failure time $T$ is an uncertain random variable with a belief reliability function $R_B(t)$. Let $\alpha$ be a real number from $(0,1)$. The system belief reliable life $T(\alpha)$ is defined as

$$T(\alpha) = \sup\{t | R_B(t) \geq \alpha\}.$$
Some belief reliability indexes

Mean time to failure (MTTF)

**Definition** (Mean time to failure)

Assume the system failure time $T$ is an uncertain random variable with a belief reliability function $R_B(t)$. The mean time to failure (MTTF) is defined as

$$MTTF = E[T] = \int_0^{\infty} \text{Ch}\{T > t\} \, dt = \int_0^{\infty} R_B(t) \, dt .$$

**Theorem**

Let $R_B(t)$ be a continuous and strictly decreasing function with respect to $t$ at which $0 < R_B(t) < R_B(0) \leq 1$ and $\lim_{t \to +\infty} R_B(t) = 0$. Then we have

$$MTTF = \int_0^1 T(\alpha) \, d\alpha .$$
Belief life variance (BLV)

**Definition** (Belief life variance)

Assume the system failure time $T$ is an uncertain random variable and the mean time to failure is MTTF. The belief life variance (BLV) is defined as

$$BLV = V[T] = E[(T - MTTF)^2].$$

**Theorem**

Let the belief reliability function be $R_B(t)$, then the BLV can be calculated by

$$BLV = \int_0^\infty R_B(MTTF + \sqrt{t}) + 1 - R_B(MTTF - \sqrt{t}) \, dt.$$
Belief reliability for uncertain systems
Belief reliability for uncertain systems

Minimal cut set theorem for uncertain system

• Uncertain system is a system only composed of uncertain components. Its belief reliability can be calculated using minimal cut set theorem.

Minimal cut set theorem

Consider a coherent uncertain system comprising \( n \) independent components with belief reliabilities \( R_{B,i}^{(U)}(t), i = 1,2, ..., n \). If the system contains \( m \) minimal cut sets \( C_1, C_2, \ldots, C_m \), then the system belief reliability is

\[
R_{B,S}(t) = \bigwedge_{1 \leq i \leq m} \bigvee_{j \in C_i} R_{B,j}^{(U)}
\]

Belief reliability for uncertain systems

Some examples

An uncertain series system has \( n \) minimal cut sets, i.e., \( C_1 = \{1\}, C_2 = \{2\}, \ldots, C_n = \{n\} \). Then the belief reliability is

\[
R_{B,S} = \min_{1 \leq i \leq n} \max_{j \in C_i} R_{B,j} = \min_{1 \leq i \leq n} R_{B,i}
\]

An uncertain parallel system only has 1 minimal cut sets, i.e., \( C_1 = \{1, 2, \ldots, n\} \). Then the belief reliability is

\[
R_{B,S} = \max_{1 \leq i \leq n} R_{B,i}
\]

An uncertain \( k \)-out-of-\( n \) system has \( \binom{n}{n-k+1} \) minimal cut sets and each set contains \( n - k + 1 \) components arbitrary chosen from the \( n \) components. Assume \( R_{B,1} \geq R_{B,2} \geq \cdots \geq R_{B,n} \), then belief reliability is

\[
R_{B,S} = R_{B,k}
\]

Belief reliability for uncertain systems

Uncertain fault tree analysis

• The belief reliability of uncertain system can be analyzed based on fault tree. The algorithm is an application of the minimal cut set theorem

Algorithm: BR analysis based on fault tree

1. Do a depth-first-search for the logic gates in the fault tree
2. For each logic gate, calculate the belief reliability for its output event:
   \[ R_{B,\text{out}} = \begin{cases} \bigwedge_{1 \leq i \leq n} R_{B,\text{in},i}, & \text{for an OR gate} \\ \bigvee_{1 \leq i \leq n} R_{B,\text{in},i}, & \text{for an AND gate} \end{cases} \]
3. \( R_{B,S} \leftarrow R_{B,\text{out},\text{TE}} \)
4. Return \( R_{B,S} \)

Belief reliability for uncertain systems

**An example:** BR analysis of the left leading edge flap of F-18

![Schematic diagram of the F-18 left leading edge flap (LLEF)](image)

**Fig.** Schematic diagram of the F-18 left leading edge flap (LLEF)

Belief reliability for uncertain systems

An example: BR analysis of the left leading edge flap of F-18

The fault tree of the F-18 LLEF

The system belief reliability is:
\[
R_{B,S} = R_{B,1} \land R_{B,2} \land R_{B,3} \land \left( \left( R_{B,5} \land R_{B,8} \right) \lor \left( R_{B,6} \land R_{B,9} \right) \right) \land \left( R_{B,4} \lor R_{B,5} \lor R_{B,6} \lor R_{B,7} \right)
\]

Fig. The fault tree of the F-18 LLEF

1 - HSA-A fail     2 - Left asymmetry control unit fail
3 - LLEF fail      4~7 - CH 1~4 fail
8 - FCC-A fail     9 - FCC-B fail

Belief reliability analysis for uncertain random systems
Simple and complex systems

Simple systems

Complex systems

Random components
Uncertain components
Random subsystem
Uncertain subsystem
Belief reliability formula for simple systems

Theorem (Simple system formula)

Assume an uncertain random system is simplified to be composed of a random subsystem with belief reliability $R_{B,R}^{(P)}(t)$ and an uncertain subsystem with belief reliability $R_{B,U}^{(U)}(t)$. If the two subsystems are connected in series, the system belief reliability will be

$$R_{B,S}(t) = R_{B,R}^{(P)}(t) \cdot R_{B,U}^{(U)}(t).$$

If the two subsystems are connected in parallel, the system belief reliability will be

$$R_{B,S}(t) = 1 - \left(1 - R_{B,R}^{(P)}(t)\right) \cdot \left(1 - R_{B,U}^{(U)}(t)\right).$$
Belief reliability formula for simple systems

Some examples

Series system

\[ R_{B,R}^{(P)}(t) \]

\[ R_{B,U}(t) \]

\[ R_{B,S}(t) = R_{B,R}^{(P)}(t) \cdot R_{B,U}(t) \]

\[ = \prod_{i=1}^{m} R_{B,i}^{(P)}(t) \cdot \bigwedge_{j=1}^{n} R_{B,j}^{(U)}(t). \]

Parallel series system

\[ R_{B,R}^{(P)}(t) \]

\[ R_{B,U}(t) \]

\[ R_{B,S}(t) = R_{B,R}^{(P)}(t) \cdot R_{B,U}(t) \]

\[ = \left( 1 - \prod_{i=1}^{m} (1 - R_{B,i}^{(P)}(t)) \right) \cdot \bigvee_{j=1}^{n} R_{B,j}^{(U)}(t). \]
Belief reliability formula for simple systems

Some examples

Parallel system

\[ R_{B,R}^{(P)}(t) \]

\[ R_{B,U}^{(U)}(t) \]

\[
R_{B,S}(t) = 1 - \left(1 - R_{B,R}^{(P)}(t)\right) \cdot \left(1 - R_{B,U}^{(U)}(t)\right) \\
= 1 - \left(\prod_{i=1}^{m} (1 - R_{B,i}^{(P)}(t))\right) \cdot \left(1 - \bigvee_{j=1}^{n} R_{B,j}^{(U)}(t)\right).
\]

Series parallel system

\[ R_{B,R}^{(P)}(t) \]

\[ R_{B,U}^{(U)}(t) \]

\[
R_{B,S}(t) = 1 - \left(1 - R_{B,R}^{(P)}(t)\right) \cdot (1 - R_{B,U}^{(U)}(t)) \\
= 1 - \left(1 - \prod_{i=1}^{m} R_{B,i}^{(P)}(t)\right) \cdot \left(1 - \bigwedge_{j=1}^{n} R_{B,j}^{(U)}(t)\right).
\]
Belief reliability formula for complex systems

Theorem (Complex system formula, Wen & Kang, 2016)

Assume an uncertain random system is a Boolean system. The system has a structure function $f$ and contains random components with belief reliabilities $R_{B,i}^{(P)}(t), i = 1, 2, \ldots, m$ and uncertain components with belief reliabilities $R_{B,j}^{(U)}(t), j = 1, 2, \ldots, n$. Then the belief reliability of the system is

$$R_{B,S}(t) = \sum_{(y_1, \ldots, y_m)\in\{0,1\}^m} \left( \prod_{i=1}^{m} \mu_i(y_i, t) \right) \cdot Z(y_1, y_2, \ldots, y_m, t),$$

where $Z(y_1, y_2, \ldots, y_m, t)$

$$Z(y_1, y_2, \ldots, y_m, t) = \begin{cases} \sup_{f(y_1, \ldots, y_m, z_1, \ldots, z_n(t))=1} \min_{1 \leq j \leq n} \nu_j(z_j, t), & \text{if } \sup_{f(y_1, \ldots, y_m, z_1, \ldots, z_n(t))=1} \min_{1 \leq j \leq n} \nu_j(z_j, t) < 0.5, \\ 1 - \sup_{f(y_1, \ldots, y_m, z_1, \ldots, z_n(t))=0} \min_{1 \leq j \leq n} \nu_j(z_j, t), & \text{if } \sup_{f(y_1, \ldots, y_m, z_1, \ldots, z_n(t))=1} \min_{1 \leq j \leq n} \nu_j(z_j, t) \geq 0.5, \end{cases}$$

$$\mu_j(y_i, t) = \begin{cases} R_{B,i}^{(P)}(t), & \text{if } y_i = 1, \ (i = 1, 2, \ldots, m), \\ 1 - R_{B,i}^{(P)}(t), & \text{if } y_i = 0, \end{cases}$$

$$\nu_j(z_j, t) = \begin{cases} R_{B,j}^{(U)}(t), & \text{if } z_j = 1, \ (j = 1, 2, \ldots, n), \\ 1 - R_{B,j}^{(U)}(t), & \text{if } z_j = 0, \end{cases}$$
A numerical case study

Table. Failure time distribution of components

<table>
<thead>
<tr>
<th>No.</th>
<th>Components type</th>
<th>Failure time distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,3,4,5</td>
<td>Random</td>
<td>$Exp(\lambda = 10^{-3} h^{-1})$</td>
</tr>
<tr>
<td>2</td>
<td>Uncertain</td>
<td>$L(500h, 3000h)$</td>
</tr>
<tr>
<td>6,7</td>
<td>Uncertain</td>
<td>$L(700h, 2700h)$</td>
</tr>
</tbody>
</table>

Figure. System belief reliability function
Belief reliability analysis for uncertain random components
The model may not precisely describe the function behavior, thus we need to add an uncertain random variable to quantify epistemic uncertainty.

\[ m = g(x_1, x_2, \cdots, x_n) \]

Parameters in the model may be uncertain because of inherent variability and the uncertainty of real working conditions. Thus they are modeled as uncertain random variables.

\[ m = g(x_1(\eta_1), x_2(\eta_2), \cdots, x_n(\eta_n)) \]
BR analysis considering parameter uncertainty

The model may not precisely describe the function behavior, thus we need to add an uncertain random variable to quantify epistemic uncertainty.

$$m = g(x_1, x_2, \ldots, x_n)$$

Parameters in the model may be uncertain because of inherent variability and the uncertainty of real working conditions. Thus they are modeled as uncertain random variables.

$$m = g(x_1(\eta_1), x_2(\eta_2), \ldots, x_n(\eta_n))$$
BR analysis with parameter uncertainty in margin model

Performance margin

**Definition (Performance margin)**

Assume the critical performance parameter of a system or a component is $p$, and its failure threshold is $p_{th}$, i.e., the system or the component will fail when $p > p_{th}$. Then the performance margin is defined as:

$$m = p_{th} - p$$

**Remark:**

1. The system or the component will be working when $m > 0$, and fail when $m < 0$.
2. Considering the parameter uncertainty of performance parameter and its threshold, there will be several cases:
   • $p$ and $p_{th}$ are both random variables
   • $p$ and $p_{th}$ are both uncertain variables
   • $p$ is a random variable and $p_{th}$ is an uncertain variable
   • $p_{th}$ is a random variable and $p$ is an uncertain variable

BR analysis with parameter uncertainty in margin model

**Case 1:** $p$ and $p_{th}$ are both uncertain variables

**Theorem 1**

Suppose the system critical performance parameter $p$ and its associated failure threshold $p_{th}$ are both uncertain variables, and their uncertainty distributions are $\Phi(x)$ and $\Psi(x)$, respectively. Then the system belief reliability will be:

$$R_B = \sup_{y \in \mathbb{R}} (\Phi(y) \land (1 - \Psi(y))).$$

**A special case**

If $p_{th}$ is a constant, then the belief reliability will be: $R_B = \Phi(p_{th})$.

---

BR analysis with parameter uncertainty in margin model

**Case 2:** $p$ is random and $p_{th}$ is uncertain

**Theorem 2**
Suppose the system critical performance parameter $p$ is a random variable with a probability distribution $\Phi(x)$, and the failure threshold $p_{th}$ is an uncertain variable with an uncertainty distribution $\Psi(x)$. Then the system belief reliability is:

$$R_B = \int_{-\infty}^{+\infty} 1 - \Psi(y) \ d\Phi(y)$$

**Case 3:** $p$ is uncertain and $p_{th}$ is random

**Theorem 3**
Suppose the system critical performance parameter $p$ is an uncertain variable with an uncertainty distribution $\Phi(x)$, and the failure threshold $p_{th}$ is a random variable with a probability distribution $\Psi(x)$. Then the system belief reliability is:

$$R_B = \int_{-\infty}^{+\infty} \Phi(y) \ d\Psi(y)$$
BR analysis with parameter uncertainty in margin model

**Case study:** Belief reliability analysis of a contact recording head

\[
V = k_s \cdot L_s \cdot W \cdot \left( \frac{L}{L_s} \right)^{1-a} \left( \frac{B}{b} \right)^{a}
\]

<table>
<thead>
<tr>
<th>Input parameters</th>
<th>Value or distribution</th>
<th>Input parameters</th>
<th>Value or distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specific wear amounts (k_s)</td>
<td>(2.55 \times 10^{-20} (m^2/N))</td>
<td>Sliding width (B)</td>
<td>(0.015 (m))</td>
</tr>
<tr>
<td>Running-in coefficient (a)</td>
<td>(0.39)</td>
<td>Contact area (A)</td>
<td>(10^{-8} (m^2))</td>
</tr>
<tr>
<td>Standard sliding distance (L_s)</td>
<td>(1000 (m))</td>
<td>Head width (b)</td>
<td>(10^{-4} (m))</td>
</tr>
<tr>
<td>Total sliding distance (L)</td>
<td>(3.6 \times 10^6 (m))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Contact load (W)</td>
<td>(W \sim \mathcal{N}(\mu = 0.7, \sigma = 0.03)(mN))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The uncertainty distribution of \(V\): \(V \sim \mathcal{N}(\mu_V = 1.8606, \sigma_V = 0.07974)(10^{-17} m^3)\)

The uncertainty distribution of \(V_{th}\) is estimated to be: \(V_{th} \sim \mathcal{L}(a = 2, b = 2.5)(10^{-17} m^3)\)

\[
R_B = \sup_{x \in \mathbb{R}} \left( \Phi_V(x) \land (1 - \Phi_{V_{th}}(x)) \right) = 0.97078
\]
BR analysis with both model and parameter uncertainties

Performance margin model

\[ m = g(x_1, x_2, \ldots, x_n) \]

Model uncertainty

The model may not precisely describe the function behavior, thus we need to add an uncertain random variable to quantify epistemic uncertainty.

\[ m = g(x_1, x_2, \ldots, x_n, E) \]

Parameter uncertainty

Parameters in the model may be uncertain because of inherent variability and the uncertainty of real working conditions. Thus they are modeled as uncertain random variables.

\[ m = g(x_1(\eta_1), x_2(\eta_2), \ldots, x_n(\eta_n)) \]
Outline

- Research Background
- Requirements Analysis
- Theoretical Framework
- Conclusion & Future
## Conclusion

- **Belief reliability**
- **Failure time**
- **Performance margin**
- **Function level**

- **Probability theory**
- **Uncertainty theory**
- **Chance theory**

- **Big data**
- **Spare data**
- **Model uncertainty**
- **Parameter uncertainty**
- **Boolean system**
- **Multi-state system**
References

Journal papers


References

Journal papers


Future

Abstract Objects

Cyber Physics Social System
Cyber Physics System
Network
Software
Hardware

Methodology

Failure/Fault Prevention
Failure/Fault Diagnosis
Failure/Fault Prognosis
Failure/Fault Control

Failurology

Recognize Failure Rules & Identify Failure Behaviors
Thank you!

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