Converting a Multi-state System into a Family of Binary State Systems

FUMIO OHI
Omohi College, Nagoya Institute of Technology
Gokiso-cho, Showa-ku, Nagoya, 466-8555, JAPAN

(Received on August 31, 2014, revised on December 16, 2014)

Abstract: In this paper we show mutual relationships between a multi-state system and a family of binary state systems and show some relationships among minimal state vectors of the systems, which give us how to construct the structure function of the multi-state system by using the structure functions of the binary state systems. But the state spaces of multi-state systems are assumed to be totally ordered set, because of simplicity for our first examination about the transformation.

Keywords: Increasing multi-state system, binary state system, totally ordered state space, minimal state vector

1. Introduction

A basic problem of reliability theory is to explain order theoretical and probabilistic relationships between a system and components consisting the system. Many studies about coherent binary state system (BSS) s have been performed, where all the state spaces are assumed to be \{0, 1\}, where 0 and 1 denote the failure and the functioning state, respectively. See the pioneering papers Mine [1], Birnbaum and Esary[2], Birnbaum, Esary and Saunder[3] and Esary and Proschan[4]. These works have been summarized by Barlow and Proschan[5]. In many practical situations, however, systems and their components could take many intermediate performance levels between the perfectly functioning and the complete failure states, and furthermore there possibly exist some states which cannot be ordered in accordance with deteriorating level, therefore a reliability model of a multi-state system (MSS) having totally or partially ordered state spaces are required for understanding and solving practical reliability problems, and some methods have been proposed to evaluate stochastic performances of MSSs in steady and transient states.

Mathematical studies of MSSs with totally ordered state spaces have been carried out by many authors, Barlow and Wu[6], Griffith[7], El-Neweihii, Proschan and Sethuraman[8], Natvig[9], Ohi and Nishida[10][11][12], Ohi[13], and so on. Huang, Zuo and Fang[14] presented an extension of consecutive \( k \)-out-of-\( n \) system, which is frequently observed in real situations, to a multi-state case. Recent works about multi-state systems are partly summarized by Natvig[15], Lisnianski and Levitin[16] and Lisnianski, Frenkel and Ding[17], where we may observe some practical applications of multi-state reliability models. Ushakov[18][19] first proposed the universal generating function method for stochastic analysis of MSSs and then Levitin[20][21][22] have intensively applied the method to MSSs of some type as series and parallel systems, showing the effectiveness of the method. Ohi[23] presented stochastic bounds for system’s stochastic performances via a modular decomposition, which is convenient for designer and analysts of real systems. However, the studies about the case of partially ordered state spaces have been recently started. Levitin [24], from a practical point of view, has proposed a new class of \( k \)-out-of-\( n \) MSSs called multi-state vector-\( k \)-out-of-\( n \) systems of which state spaces are sets of vectors and so a particular type of partially ordered sets. Yu, Koren and Guo [25] presented a model...
of a reliability system having partially ordered state spaces. Aiming at building a theoretical framework of MSSs, a mathematical generalization to the case of partially ordered state spaces has been given by Ohi[26][27]. Ohi[28] showed, using a modular decomposition, stochastic bounds for the probability that a state of a MSS is greater than or equal to s, where all the state spaces were assumed to be partially ordered sets, but the proof was not presented.

These methods, however, need perfect knowledge about the correspondences between the states of a system and those of components, or partial knowledge about a structure function as minimal or maximal state vectors. These kinds of state vectors implicitly determine the structure function. For a BSS, we may have such knowledge by a fault tree analysis (FTA) method which determines minimal path and cut sets or equivalently minimal cut and path vectors of the system and then implicitly the structure function. But for a MSS, although the concept of system is clearly defined, we have not yet had such a convenient tool like FTA to determine a structure function.

In this paper we show some relationships between a MSS and a family of BSSs. The family is not arbitrary but restricted to be “adaptive”, a concept newly proposed in this paper. By this relation, minimal path vectors of the family of BSSs are related to the minimal state vectors of a MSS, and then the structure function of the MSS may be determined via the structure functions of the BSSs practically determined by FTA. But our examinations presented in this paper are the first step aimed at the determination of the structure functions of MSSs, and some problems remains to be solved in near future.

This paper newly proposes two concepts, “adaptive” and “perfectly adaptive,” in the Section 3. Then we relate minimal path vectors of an adaptive family of BSSs to minimal state vectors of a MSS in the Section 4. Section 5 is devoted to how the normal and relative properties of a MSS are reflected in an adaptive family of BSSs. Throughout this paper, a MSS is basically assumed to be increasing. Some definitions of reliability theoretic concepts needed in this paper are presented in Section 2.

Notation

In this paper we use the following notation. Finite sets \( C = \{1,2,\cdots,n\} \), \( \Omega_i (i \in C) \) and \( S \) are respectively the set of the components, the state space of the \( i \)-th component and the state space of the system. The state spaces of which cardinal numbers are not necessarily the same are assumed to be totally ordered sets. \( \varphi \) is a mapping from the product ordered set \( \Omega_c = \prod_{i \in C} \Omega_i = \prod_{i=1}^n \Omega_i \) to \( S \), called a structure function of the system. The precise definition of a MSS is presented in Definition 2

1. An element \( x \in \Omega_c \) is precisely written as \( x = (x_1,x_2,\cdots,x_n), x_i \in \Omega_i (i = 1,\cdots,n) \). \((k,x) (i \in C) \) denotes that the \( i \)-th element of \( x \) is \( k \).
2. The orders on \( \Omega_i (i \in C) \) and \( S \) are denoted by a common symbol \( \leq \), which is also used for the product order on \( \Omega_c \), so for any \( x \) and \( y \) of \( \Omega_c \), \( x \leq y \) is equivalent to \( x_i \leq y_i \) for every \( i \in C \). Furthermore, \( x < y \) means that for every \( i \in C \), \( x_i \leq y_i \) and for some \( j \in C \), \( x_j \neq y_j \).
3. For any state \( s \in S \),
\[
\varphi^{-1}(s \leq) = \{ x \in \Omega_c | s \leq \varphi(x) \}, \quad \varphi^{-1}(\leq s) = \{ x \in \Omega_c | \varphi(x) \leq s \},
\]
\[
\varphi^{-1}(s) = \{ x \in \Omega_c | \varphi(x) = s \}.
\]
4. \( MI(W) \) and \( MA(W) \) generally denote the sets of all the minimal and maximal elements of a finite ordered set \( W \), respectively. An element \( x \) of \( W \) is called a minimal (maximal) element when there exists no element \( y \) of \( W \) such that \( y < x \) (\( x < y \)). We simply use the following abbreviations for a system \( \varphi \).
\[ M_\varphi(s \leq) = M_\varphi(\varphi^{-1}(s \leq)) \] 
\[ M_\varphi(s) = M_\varphi(\varphi^{-1}(s)) \]

5. A finite ordered set \( W \) is called an increasing (a decreasing) set when \( x < y \) (or \( x > y \)) imply \( y \in W \).

6. The symbol \( \setminus \) denotes the difference between two sets \( A \) and \( B \) as \( A \setminus B = \{ x \mid x \in A, x \notin B \} \).

7. \( B_i \) is an binary set \( \{0,1\} \) relating to the state \( k \) of the component \( i \). Precise explanation is given in the Section 3.

8. \( \varphi \cap \subseteq \) and \( \varphi \cup \) are the two types of BSSs given by decomposing a MSS \( \varphi \).

9. MSS and BSS are the abbreviations of "multi-state system" and "binary state system", respectively.

2. Preliminary Definitions

**Definition 1:** (Definition of a MSS) A MSS composed of \( n \) components is a triplet \( (\Omega_C, S, \varphi) \) satisfying the following conditions.

(i) \( C = \{1, \cdots , n\} \) is the set of integers from 1 to \( n \), where each number is the index of each unit.

(ii) \( \Omega_i (i \in C) \) is a finite totally ordered set and assumed to be \{0,1, \cdots , N_i\} without loss of generality. \( S \) is also a finite totally ordered set and is assumed to be \{0,1, \cdots , N\}.

(iii) \( \Omega_C = \prod_{i=1}^{n} \Omega_i \) is the product ordered set of \( \Omega_i (i \in C) \). An element \( x = (x_1, \cdots , x_n) \in \Omega_C \) is called a state vector.

(iv) \( \varphi \) is a surjection from \( \Omega_C \) to \( S \), which is called a structure function.

A MSS \( (\Omega_C, S, \varphi) \) is simply called a MSS \( \varphi \) when there is no confusion.

**Definition 2:** (Increasing Property) A MSS \( \varphi \) is called increasing, when \( \varphi(x) \leq \varphi(y) \) holds for every \( x \) and \( y \in \Omega_C \) such that \( x \leq y \). The increasing property means that the state of the system is to be better for improved components.

The following assertion about an increasing MSS \( (\Omega_C, S, \varphi) \) is a generalization of the assertion that the structure function of a BSS is uniquely determined by the set of minimal path or cut sets, or equivalently, by minimal path or cut vectors. See Barlow and Proschan [5].

\[ \varphi(x) = s \Leftrightarrow \exists a \in M_\varphi(s \leq), x \geq a, \] 
\[ \forall t \text{ such that } t > s, \forall b \in M_{\varphi}(t \leq), x \text{ is not greater than or not equal to } b \] 
\[ \Leftrightarrow s = \max \{ t \mid \exists a \in M_{\varphi}(t \leq), x \geq a \} \]
\[ \varphi(x) \geq s \Leftrightarrow \exists a \in M_{\varphi}(s \leq), x \geq a, \]

which shows us that an increasing MSS is uniquely determined by the family of the sets of the minimal state vectors, \( \{ M_{\varphi}(s \leq) \}_{s \in S} \). Similarly, considering the dual order, an increasing MSS is also uniquely determined by the family of the sets of the maximal state vectors, \( \{ M_{\varphi}(s \leq) \}_{s \in S} \).

The next example from Ohi[23] demonstrates how a structure function is uniquely determined by minimal state vectors, and is used in the sequel examinations.

**Example 1:** An increasing MSS in this example is composed of three components having the following state spaces.

\( \Omega_1 = \Omega_2 = \{0,1,2\}, \Omega_3 = \{0,1\}, S = \{0,1,2\}. \)

The structure function \( \varphi \) of the system is uniquely determined by
\[ M_\varphi(1 \cong) = \{(0,1,1), (1,1,0), (2,0,1)\}, \quad M_\varphi(2 \cong) = \{(1,2,1), (2,1,0), (2,0,1)\}. \]

For a state vector \((1,2,0)\), which is greater than a minimal state vector \((1,1,0)\) of \(M_\varphi(1 \cong)\) and is not greater than any state vector of \(M_\varphi(2 \cong)\), then \(\varphi(1,2,0) = 1\).

For this system, we have
\[ M_\varphi(1) = \{(0,1,1), (1,1,0)\}, \quad M_\varphi(2) = \{(1,2,1), (2,1,0), (2,0,1)\}, \]
and then \(M_\varphi(1)\) differs from \(M_\varphi(1 \cong)\).

Following Ohi[23], we present a definition of normal property which is crucial for deriving stochastic bounds for reliability of a multi-state system via a modular decomposition.

**Definition 3:** (Normal Property) (i) A MSS \(\varphi\) is called minimally normal when
\[ \forall s, \forall t \in S (s \neq t), \quad M_\varphi(s \cong) \cap M_\varphi(t \cong) = \varnothing. \]
(ii) A MSS \(\varphi\) is called maximally normal when
\[ \forall s, \forall t \in S (s \neq t), \quad M_\varphi(\leq s) \cap M_\varphi(\leq t) = \varnothing. \]

**Example 2:** For the MSS of the Example 1 which is not minimally normal, it is easily verified that for a state vector \((1,0,1)\), \(\varphi(1,0,1) = 0\) and \(\varphi(2,0,1) = 2\), meaning that the system’s state changes from 0 to 2 by the state change of the component 1 from 1 to 2.

**Definition 4:** (Relevant Property) (i) The component \(i \in C\) is said to be relevant when the following is satisfied.
\[ \forall k, \forall l \in \Omega_i (k \neq l), \exists (k_i, x), \exists (l_i, x), \varphi(k_i, x) \neq \varphi(l_i, x). \]
(ii) A MSS \(\varphi\) is called relevant when every component is relevant.

The condition (i) has essentially no practical restriction on the system, since if a component \(i \in C\) does not satisfy the condition, we have
\[ \exists k, \exists l \in \Omega_i (k \neq l), \forall (k_i, x), \forall (l_i, x), \varphi(k_i, x) = \varphi(l_i, x), \]
which means that the states \(k\) and \(l\) equivalently contribute to the system and then may be merged into one state.

**Definition 5:** (Binary-state system) A MSS \((\prod_{i \in C} \Omega_i, S, \varphi)\) is called a BSS, when the state spaces are assumed to be \(\Omega_i = \{0, 1\} (i \in C\) and \(S = \{0, 1\}\).

For a BSS \(\varphi, M_\varphi(1 \cong)\) and \(M_\varphi(0 \cong)\) are well known to be the sets of minimal path and cut vectors, respectively.

### 3. Adaptive Family of Binary State Systems

In this section, we consider a MSS \((\prod_{i \in C} \Omega_i, S, \varphi)\). For \(i \in C\) and \(k \in \Omega_i \setminus \{0\}\), \(B_i^k\) and \(B_i^0\) are sets of two elements 0 and 1. Orders on these binary sets are defined to be \(0 < 1\).

On the product sets \(\prod_{k \in \Omega_i \setminus \{0\}} B_i^k\) \((i \in C)\) and \(\prod_{k \in \Omega_i \setminus \{0\}} B_i^k\), we consider the product orders which are denoted commonly by \(\cong\). The product set \(\prod_{k \in \Omega_i \setminus \{0\}} B_i^k\) is used to embed the state space \(\Omega_i\) of the component \(i\) by corresponding \(x_i \in \Omega_i\) to \((1, \cdots, 1, 0, \cdots, 0) \in \prod_{k \in \Omega_i \setminus \{0\}} B_i^k\). Of course, it seems to be natural to correspond 0 and 1 of \(B_i^k\) to the nonoccurrence and the occurrence of the state \(x_i\) of the component \(i\), respectively. This correspondence does not assure for the BSS defined in the sequel to be increase with respect to the product order on \(\prod_{k \in \Omega_i \setminus \{0\}} B_i^k\), which is usually assumed in the theory of BSSs.

In the sequel we use the following notation. For \(x_i \in \Omega_i, 1 \leq i \leq n,\)
\[ 1(x_i) = (1, \cdots, 1, 0, \cdots, 0) \in \prod_{k \in \Omega_i \setminus \{0\}} B_i^k, \]
\[ 1(x_i) = (0,\ldots,0) \text{ if } x_i = 0, \]
\[ j_i = \{1(x_i) | 1 \leq x_i \leq N_i \} \cup \{(0,\ldots,0)\}. \]

Furthermore a symbol \(1(x)\) \((x \in \Omega_c)\) denotes\[ 1(x) = (1(x_1), 1(x_2), \ldots, 1(x_n)). \]

\(J_i\) is order isomorphic to \(\Omega_i\) \((i \in C)\) and \(\prod_{i \in C} J_i\) is also order isomorphic to \(\Omega_C\). For \(d = (d_1,\ldots,d_n) \in \prod_{i \in C} \prod_{k \in \{0\}} B_k^k\), where \(d_i = (d_1^i,\ldots,d_n^i) \in \prod_{k \in \{0\}} B_k^k\), \(i \in C\) and \(d_k^i \in B_k^k\), \(h\) is defined to be\[ h(d_i) = \max \{ k | d_k^i = 1 \} , \quad h(d) = (h(d_1),\ldots,h(d_n)). \]

Example 3: (i) For the MSS of the Example 1, we have\[ C = \{1,2,3\}, \Omega_1 = \Omega_2 = \{0,1,2\}, \Omega_3 = \{0,1,2\}, S = \{0,1,2\}, \]
\[ B_1^1 = B_2^1 = \{0,1\}, B_2^2 = \{0,1\}, B_3^3 = \{0,1\}, \]
\[ J_1 = \{(0,0),(1,0),(1,1)\}, J_2 = \{(0,0),(1,0),(1,1)\}, J_3 = \{(0,1)\}. \]

For \(x = (1,2,1) \in \Omega_1 \times \Omega_2 \times \Omega_3\) as an example,\[ 1(x) = (1(1),1(2),1(1)) = (1,0,1,1,1). \]

The product set of \(J_i (i = 1,2,3)\) is\[ \prod_{i \in C} J_i = \{(0,0,0,0,0),(0,0,0,0,1),(0,0,0,1,0),(0,0,0,1,1),(0,0,1,0,0),\]
\[ (0,0,1,0,1),(0,0,0,0,0),(0,0,1,0,0),(0,1,0,0,0),(0,1,0,0,1),(0,1,0,1,0),(0,1,0,1,1),\]
\[ (1,0,1,0,0),(1,0,1,0,1),(1,0,1,1,0),(1,0,1,1,1),(1,1,0,0,0),(1,1,0,0,1),(1,1,0,1,0),\]
\[ (1,1,0,1,1),(1,1,1,0,0),(1,1,1,0,1),(1,1,1,1,1)\}. \]

(ii) For \(d_1 = (0,1) \in B_1^1 \times B_2^2\), \(d_2 = (1,0) \in B_2^2 \times B_2^2\), \(d_3 = (1) \in B_3^3\), we have\[ h(d_1) = 2 , h(d_2) = 1 , h(d_3) = 1 , \]
\[ d = (d_1,d_2,d_3) = (0,1,0,1,0) \in B_3^3 \times B_2^2 \times B_2^2 \times B_3^3 , \]
and then\[ h(d) = (h(d_1),h(d_2),h(d_3)) = (2,1,1) \in \Omega_1 \times \Omega_2 \times \Omega_3 . \]

Definition 6: (Adaptive family of BSSs) A family of increasing BSSs\[ \left( \prod_{i \in C} \prod_{k \in \{0\}} B_k^k , B , \vec{\phi}_{\leq t} \right) , t \in S \]
is called adaptive, which is abbreviated as \(\{ \vec{\phi}_{\leq t} \}_{t \in S}\), when the following condition holds:\[ \forall d \in \prod_{i \in C} J_i , \vec{\phi}_{\leq t}(d) = 1 \Rightarrow \forall s(\leq t) , \vec{\phi}_{s\leq t}(d) = 1 . \]

Definition 7: (Adaptive to a MSS) Let \(\left( \prod_{i = 1}^N \Omega_i , S , \varphi \right)\) be an increasing MSS. An adaptive family of increasing BSSs \(\left( \prod_{i = 1}^N \prod_{k \in \{0\}} B_k^k , B , \vec{\phi}_{\leq t} \right) \) is said to be adaptive to the MSS \(\varphi\), when for \(x \in \prod_{i = 1}^N \Omega_i\), \(\varphi(x) \cong t \Leftrightarrow \vec{\phi}_{t\leq t}(1(x)) = 1\).

Example 4: (i) For the MSS of Example 1, we define \(\tilde{A}_1\) and \(\tilde{A}_2\), corresponding to \(M_l\varphi(1 \leq s)\) and \(M_l\varphi(2 \leq s)\), respectively, as\[ \tilde{A}_1 = \{(0,0,1,0,1),(1,0,1,0,0),(1,1,0,0,1)\} \subseteq \prod_{i = 1}^3 \prod_{k \in \{0\}} B_k^k , \]
\[ \tilde{A}_2 = \{(1,0,1,1,1),(1,1,1,0,0),(1,1,0,0,1)\} \subseteq \prod_{i = 1}^3 \prod_{k \in \{0\}} B_k^k . \]

Then defining \(\vec{\phi}_{1\leq s}\) and \(\vec{\phi}_{2\leq s}\) as
it is easily verified that \( \{ \bar{\varphi}_1, \bar{\varphi}_2 \} \) is adaptive to the MSS \( \varphi \).

(ii) If we define

\[
A_1 = \{(0,0,0,0,0), (1,0,1,0,0), (1,1,0,0,1), (1,0,0,1,0)\}, \quad A_2 = \bar{A}_2,
\]

and

\[
\bar{\nu}_1(\ell) = 1, \text{ if } \exists a \in A_1, \ell \geq a, \quad \bar{\nu}_2(\ell) = 1, \text{ if } \exists a \in A_2, \ell \geq a,
\]

then \( \{ \bar{\nu}_1, \bar{\nu}_2 \} \) is also adaptive to the MSS \( \varphi \).

These examples, (i) and (ii), show us that we have some degree of freedom for determining a family of BSSs adaptive to a MSS. Notice

\[
\forall \ell \in \prod_{i \in \Omega_i} \prod_{j \in \Omega_i \setminus \{0\}} B_i^k, \quad \bar{\varphi}_1(\ell) \leq \bar{\nu}_1(\ell), \quad \bar{\varphi}_2(\ell) \leq \bar{\nu}_2(\ell).
\]

which suggest us a kind of minimum property of the family of (i). This property is generally formalized in the following Definition 8 and Proposition 1.

**Remark 1:** When a family of increasing BSSs \( \{ \tilde{\varphi}_t \}_{t \in S} \), which is adaptive to the MSS \( \varphi \), is given, the condition of Definition 7 is equivalent to

\[
\varphi(x) = t \iff \tilde{\varphi}_t(1(x)) = 1 \quad \text{and} \quad \bar{\varphi}_{t+1}(1(x)) = 0.
\]

Then using \( \{ \tilde{\varphi}_t \}_{t \in S} \), we may define a family of BSSs \( \{ \tilde{\varphi}_t \}_{t \in S} \) as

\[
\forall \ell \in \prod_{i \in \Omega_i} \prod_{k \in \Omega_i \setminus \{0\}} B_i^k, \quad \tilde{\varphi}_t(\ell) = \begin{cases} 1, & \ell \in A_t, \\
0, & \text{otherwise.}
\end{cases}
\]

Hence it is easily verified that for every \( x \in \Omega_C \), the family satisfies

\[
\varphi(x) = t \iff \tilde{\varphi}_t(1(x)) = 1, \quad \text{(1)}
\]

\[
\tilde{\varphi}_t(1(x)) = 1 \Rightarrow \forall s \neq t, \tilde{\varphi}_s(1(x)) = 0, \quad \text{(2)}
\]

which means that \( \tilde{\varphi}_t \) is the indicator function of whether \( \varphi(x) = t \) holds or not, but is no longer an increasing function with respect to the product order.

On the other hand, when a family of BSSs \( \{ \tilde{\varphi}_t \}_{t \in S} \), satisfying (1) and (2) which is temporally called adaptive II, is first given for a MSS \( \varphi \), we may define \( \{ \tilde{\varphi}_t \}_{t \in S} \) adaptive to the MSS as

\[
\ell \in \prod_{i \in \Omega_i} \prod_{k \in \Omega_i \setminus \{0\}} B_i^k, \quad \tilde{\varphi}_t(\ell) = \max_{s \leq t} \tilde{\varphi}_s(\ell).
\]

So we may alternatively use adaptive and adaptive II families of BSSs for a MSS.

4. **Minimal State Vectors of a MSS and an Adaptive Family of BSSs**

In this section we examine a relationship between the minimal state vectors of a MSS \( \varphi \) and those of a family of BSSs \( \{ \tilde{\varphi}_t \}_{t \in S} \) adaptive to \( \varphi \). Since for \( x \in M_{\varphi}(t) \), \( \varphi(x) = t \) holds, then we have by Remark 1

\[
\tilde{\varphi}_t(1(x)) = 1, \quad \tilde{\varphi}_{t+1}(1(x)) = 0.
\]

On the other hand for \( y \in \Omega_C \) such that \( y \leq x \) and \( y \neq x \), \( \varphi(y) < t \) follows and then we have \( \tilde{\varphi}_{t+1}(1(y)) = 0 \) also by Remark 1. Hence when the domain of \( \tilde{\varphi}_t \) is restricted to \( \prod_{i \in \Omega_i} \tilde{\varphi}_i \), for \( x \in M_{\varphi}(t) \), \( 1(x) \) is a minimal element of \( \bar{\varphi}_t^{-1}(1) \). But when \( \prod_{i \in \Omega_i} \tilde{\varphi}_i^{-1} \) is chosen to be the domain, \( 1(x) \) is not necessarily minimal. First we define a special adaptive property having a kind of minimum property, and then we examine minimal state vectors in a general case.

**Definition 8:** (Perfectly adaptive to a multi-state system) Let \( (\Omega_C, S, \varphi) \) be an increasing MSS. A family of BSSs \( \{ \tilde{\varphi}_t \}_{t \in S} \) adaptive to the MSS is called perfect, when the following equality holds.

\[
M_{\varphi}(1) = \{ 1(x) \mid x \in M_{\varphi}(t) \}.
\]
Proposition 1: Let \((\Omega, S, \varphi)\) be an increasing MSS. When a family of BSSs \(\{\varphi_{t \leq s}\}_{t \in S}\) is perfectly adaptive to the MSS and a family \(\{\varphi_{t \leq s}\}_{t \in S}\) is adaptive to the MSS, we have
\[
\forall t \in S, \forall z \in \prod_{t \leq s} \prod_{t \in \{0, 1\}^{|t|}} B_t^e, \quad \varphi_{t \leq s}(z) \subseteq \varphi_{t \leq s}(z),
\]
which means that the perfectly adaptive family is the minimum in the class of families adaptive to the MSS \(\varphi\).

Proof: We have by the condition
\[
\{ y \mid \varphi_{t \leq s}(y) = 1 \} = \{ y \mid \exists x \in M I(\varphi^{-1}(t \leq s)), 1(x) \leq y \} \subseteq \{ y \mid \varphi_{t \leq s}(y) = 1 \},
\]
from which the minimum property is clear.

Example 5: \(\{\varphi_{t \leq s}, \varphi_{t \leq s}\}\) of the Example 4 is perfectly adaptive, and \(\{\varphi_{t \leq s}, \varphi_{t \leq s}\}\) is adaptive but not perfect.

Proposition 2: Let \((\Omega, S, \varphi)\) be an increasing MSS and \(\{\varphi_{t \leq s}\}_{t \in S}\) be adaptive (not necessarily perfect) to this MSS. Then for every \(t \in S\), we have
\[
M I_\varphi(t \leq s) = \{ h(d) \mid d \in M I_{\varphi_{t \leq s}}(1) \}.
\]

Proof: (the left hand side \(\subseteq\) the right hand side) For \(x \in M I_\varphi(t \leq s), \varphi(x) \subseteq t\) holds.

Thus, we have
\[
\exists h(e) \in M I_{\varphi_{t \leq s}}(1), 1(x) = (1(x_1), \ldots, 1(x_n)) \subseteq d,
\]
which means that \(x_i \geq h(d_i) = \max \{ k \mid d_i \subseteq 1 \}\). Then for this \(d\),
\[
x \geq h(d) = (h(d_1), \ldots, h(d_n)),
\]
and from the adaptability of \(\varphi_{t \leq s}\), \(\varphi_{t \leq s}(1(h(d_1)), \ldots, 1(h(d_n))) \subseteq \varphi_{t \leq s}(d) = 1\). Then by the minimal property of \(x, x = h(d) = (h(d_1), \ldots, h(d_n))\) holds and hence we have
\[
x \in \{ h(d) \mid d \in M I_{\varphi_{t \leq s}}(1) \}.
\]

Thus
\[
\exists h(e) \in M I_{\varphi_{t \leq s}}(1), x \subseteq h(e).
\]

Since \(1(x) \subseteq e, x = h(e)\) holds by an examination similar to the above one, and then the inclusion relation is proved.

Proof (the left hand side \(\supseteq\) the right hand side) Suppose
\[
h(d) \in M I_{\varphi_{t \leq s}}(1). \quad (3)
\]
Then from the adaptability property to the MSS, we have \(\varphi(h(d)) \preceq t\).

If \(h(d) \not\in M I_\varphi(t \leq s)\), then
\[
\exists a = (a_1, \ldots, a_n) \in M I_\varphi(t \leq s), h(d) \geq a, h(d) \neq a
\]
which means
\[
\exists i \in C, h(d_i) > a_i. \quad (4)
\]

For this \(a\),
\[
1(h(d)) \equiv 1(a), 1(h(d)) \neq 1(a), \varphi_{t \leq s}(1(a)) = 1
\]
and
\[
\exists d' \in M I_{\varphi_{t \leq s}}(1), 1(h(d)) \equiv 1(a) \geq d', 1(h(d)) \neq 1(a).
\]

Thus we have
\[
h(d) \geq h(d'), h(d) > h(d'),
\]
where \(i\) satisfies the inequality of (4). These inequalities contradict to the minimal property of \(h(d)\) of (3) and hence we have \(h(d) \in M I_\varphi(t \leq s)\).
Let \( \{ \tilde{\phi}_t \}_{t \in S} \) be adaptive II to an increasing MSS \( \phi \). Since \( \tilde{\phi}_t (t \in S) \) is not necessarily increasing, the following equality does not necessarily hold.

\[
MI_{\phi}(t) = MI \{ h(d) \mid d \in MI_{\tilde{\phi}_t}(1) \}.
\] (5)

But for every \( t \in S \), we have the following relation on \( \prod_{i \in C} I_i \):

for \( d_1, d_2 \in \prod_{i \in C} I_i \) such that \( d_1 \leq d_2 \),

\[
\tilde{\phi}_t(d_1) = \tilde{\phi}_t(d_2) = 1 \implies \forall d \in \prod_{i \in C} I_i \text{ such that } d_1 \leq d \leq d_2, \tilde{\phi}_t(d) = 1,
\]

since the family is adaptive II. Amplifying this property, we call \( \{ \tilde{\phi}_t \}_{t \in S} \) to be local constant, when

\[
\forall t \in S, \forall d_1, \forall d_2 \in \prod_{i \in C} I_i \text{ such that } d_1 \leq d_2, \tilde{\phi}_t(d_1) = \tilde{\phi}_t(d_2) = 1 \implies \forall d \in \prod_{i \in C} I_i \text{ such that } d_1 \leq d \leq d_2, \tilde{\phi}_t(d) = 1
\]

holds, and then we have the next proposition.

**Proposition 3:** Suppose \( \{ \tilde{\phi}_t \}_{t \in S} \) to be local constant and adaptive II to an increasing MSS \( \phi \). Then the equality relation (5) holds.

The condition of the local constant is not a severe restriction on adaptive II families, since an adaptive II family for an increasing MSS automatically satisfy the local constant property on \( \prod_{i \in C} I_i \) which is order isomorphic to \( \Omega_C \).

5. Normal and Relevant Properties

Let \( (\Omega_C, S, \phi) \) be an increasing MSS and \( \{ \tilde{\phi}_t \}_{t \in S} \) be adaptive. From Proposition 2 and the definition of the minimally normal, the next Proposition is clear.

**Proposition 4:** The system \( \phi \) is minimally normal if and only if

\[
\forall s, t \in S \text{ such that } s \neq t, \quad MI \{ h(d) \mid d \in MI_{\tilde{\phi}_s}(1) \} \cap MI \{ h(d) \mid d \in MI_{\tilde{\phi}_t}(1) \} = \phi.
\]

Since a state \( k \in \Omega_i \) of the component \( i \) corresponds to \( 1(k) \in I_i \), the relevant property of the MSS is expressed by \( \{ \tilde{\phi}_t \}_{t \in S} \) defined from \( \{ \tilde{\phi}_t \}_{t \in S} \) as the following.

**Proposition 5:** The component \( i \in C \) is relevant if one only if

\[
\forall k, \forall l \in \Omega_i \text{ such that } k \neq l, \quad \exists s \in S, \exists t \in S \text{ such that } s \neq t, \quad \forall j \in C \text{ such that } j \neq i, \exists d_j \in I_j,
\]

\[
\tilde{\phi}_t(d_1, \ldots, d_{i-1}, 1(k), d_{i+1}, \ldots, d_n) = 1, \tilde{\phi}_t(d_1, \ldots, d_{i-1}, 1(l), d_{i+1}, \ldots, d_n) = 1.
\]

6. Conclusions

In this paper we have proposed two kinds of families of BSSs adaptive to a MSS, one is defined in Definition 3 and other is given in Remark 3, which we have used to relate a MSS to a family of BSSs, especially by relationships among minimal path vectors of the family and the MSS. Although we have treated only minimal state vectors, it is easy to state our propositions in term of the maximal state vectors by the concept of dual order. When we develop a theory of multi-state FTA by utilizing the usual FTA for BSSs, these adaptive properties are thought to play important roles, but of which proof remains to be a future work. Furthermore we have shown how normal and relevant properties of a MSS are reflected in a adaptive family of BSSs.
References


**Fumio Ohi** received his Doctor of Engineering degree from Osaka University, Japan, in 1981. He joined Faculty of Engineering, Osaka University, as Research Assistant in 1978 and moved to Aichi Institute of Technology, Japan as Associate Professor in 1989. He has been working in Nagoya Institute of Technology (NIT), Japan, since 1995, and is currently Full Professor in the Department of Scientific and Engineering Simulation, Graduating School of Engineering, NIT. His research interests are fundamental theory of informatics including system reliability theory, classification problem of cellular automata, fractal analysis method of time series data, and multi-agent simulation of emergency evacuation flow of pedestrians.