An Inspection Optimization Model Based on a Three-stage Failure Process

RUIFENG YANG1, FEI ZHAO2, JIANSHE KANG1, and XINGHUI ZHANG1*

1Mechanical Engineering College, Shijiazhuang, China
2Dongling School of Economics and Management, University of Science and Technology Beijing, Beijing, China

(Received on April 22, 2014, revised on August 7, 2014 and September 7, 2014)

Abstract: Inspections are common activities in most preventive maintenance (PM) programs. The models for optimizing the inspection interval using the two-stage delay time have been presented by many researchers. However, the three-stage failure process introduced by Wang is closer to reality corresponding to the actual industrial applications. When the minor defective stage is identified at an inspection, the inspection interval is halved. However, whether this measure is optimal is not explained. In order to solve this problem, an inspection optimization model is proposed to minimize the expected cost per unit time with the inspection interval and shortening proportion of the inspection interval after identifying the minor defective stage as the decision variables. A numerical example is presented to illustrate the applicability of the proposed model.

Keywords: delay time, three-stage failure process, inspection, PM

1. Introduction

Inspection is always a necessary activity for preventive maintenance (PM), as it provides information on the status of the system. Therefore, the determination of inspection intervals is one of the key decisions. Many researchers have developed several models to optimize the inspection interval under various modelling scenarios [1], especially the delay time model (DTM). Delay time concept was first introduced by Christer to justify the necessity of inspection activities, which defines the failure process as a two-stage process, namely normal stage and delay time stage [2]. The normal stage is from new to an initial point that a defect can be firstly identified by an inspection and the delay time stage is from this initial point to failure [3]. Delay time models explicitly establish the relationship between the numbers of failures and the inspection interval and some successful case studies have been finished and applied in industrial applications [4, 5].

However, Wang [6] extends the two-stage failure process to three-stage failure process in accordance with the actual industrial phenomena that the state of system is described by a three color scheme, e.g., green, yellow and red. Based on the three-stage failure process, the states of system before failure are normal, minor defective, severe defective respectively. This further dividing provides more modeling options and is a step closer to reality. In the work by Wang [6], when the minor defective stage is identified by inspection, it is assumed that halving the current inspection interval for the subsequent inspections is a better option than maintenance immediately. But whether the policy of halving the current inspection interval is optimal is not illustrated. Therefore, in this paper, we assume the proportion of shortening the inspection interval when the minor defective stage is identified as a decision variable to be optimized. Another decision variable is the inspection interval. The objective is to minimize the expected cost per unit time.
2. Modelling Assumptions and Notation

For a modelling purpose, some assumptions are presented firstly as follows:

(1) Consider a single component system only subject to a single failure mode.

(2) The failure process is divided into three stages: normal stage $X_1$, minor defective stage $X_2$ and severe defective stage $X_3$. These three stages are assumed to be independent.

(3) Inspections are perfect so that the state of system can be identified explicitly.

(4) If the system is in the normal stage, do nothing. Once the system is found in the minor defective stage by an inspection, we assume the subsequent inspection interval is shortened to be $1/k$ of the current interval, where $k$ is viewed as a decision variable and $k>1$. However, if the system is identified to be in the severe defective stage, it is always repaired immediately.

(5) Failure can be observed immediately and replacement is always carried out at once.

(6) Repair or replacement is regarded as renewing the system, though replacement may be the only option for a single component system.

The following notation is used in subsequent modeling:

- $X_n$: random variable representing the duration of the $n$th stage, $n=1, 2$ and 3
- $f_{X_n}(x)$, $f_{X_n}(y)$: probability density function (pdf) of $X_1$, $X_2$
- $f_{X_i}(z)$, $F_{X_i}(z)$: pdf and cumulative distribution function (cdf) of $X_3$
- $t$: inspection interval
- $k$: shortening proportion of inspection interval at the time of identifying the minor defective stage
- $T_f$: random failure time
- $T_p$: random time of an inspection renewal due to the identification of the severe defective stage by an inspection
- $C_e$, $C_f$: average cost per inspection, per failure
- $C_p$: average cost caused by an inspection renewal

3. The Proposed Cost Model

The cost model is formulated in this section using the renewal theorem to optimize the inspection interval and the shortening proportion of the inspection interval once the minor defective stage is identified [3].

3.1 The Expected Renewal Cycle Cost

(1) The expected cost of a failure renewal

1) The system fails in $((m-1)t, mt)$ before any defective stage is found. So the minor and severe defective stage must happen in the same interval, see Figure 1. The probability of such a failure renewal can be given as

$$P((m-1)t < T_f < mt) = P((m-1)t < X_1 < mt, 0 < X_2 < mt - X_1, 0 < X_3 < mt - X_1 - X_2)$$

$$= \int_0^{mt} f_{X_1}(x) \int_0^{mt-x} f_{X_2}(y) F_{X_3}(mt-x-y) dy dx$$

(1)

2) The system fails in $(mt+(i-1)t/k, mt+it/k)$ after the minor defective stage is first found by inspection at $mt$. It is noted that the minor defective stage must occur within $((m-1)t, mt)$ and end in $(mt+(i-1)t/k, mt+it/k)$ from assumption (3), and the severe defective stage deteriorates to a failure in $(mt+(i-1)t/k, mt+it/k)$ ultimately, as shown in Figure 2. The occurrence probability of such a failure event is
An Inspection Optimization Model Based on a Three-stage Failure Process

\[ P(mt + (i - 1)\frac{t}{k} < T_f < mt + i\frac{t}{k}) \]
\[ = P((m-1)\mu < X_1 < mt, mt + (i-1)\mu/k - X_1 < mt + i\mu/k - X_1, \]
\[ 0 < X_1 < mt + i\mu/k - X_1 - X_2) \]
\[ = \int_{(mt-1)\mu}^{mt} f_{X_1}(x) \int_{mt+(i-1)\mu/k-x}^{mt+i\mu/k-x} f_{X_2}(y) F_{X_1}(mt + i\mu/k - x - y)dydx \]

\textbf{Figure 1: The System fails before any Defective Stage is identified}

\[ P(mt + i\mu/k < T_f < mt + (i+1)\mu/k) \]
\[ = P((m-1)\mu < X_1 < mt, mt + i\mu/k - X_1 < X_2 < mt + (i+1)\mu/k - X_1, \]
\[ 0 < X_1 < mt + i\mu/k - X_1 - X_2) \]
\[ = \int_{(mt-1)\mu}^{mt} f_{X_1}(x) \int_{mt+i\mu/k-x}^{mt+(i+1)\mu/k-x} f_{X_2}(y) F_{X_1}(mt + i\mu/k - x - y)dydx \]

\textbf{Figure 2: The System fails after identifying the Minor Defective Stage}

Therefore, the expected cost due to a failure renewal can be derived as
\[ E(C_r(t)) = \sum_{m=1}^{\infty} \left( C_r + (m-1)C_r \right) \cdot P((m-1)\mu < T_f < mt) + \sum_{i=1}^{\infty} \left( C_r + (m+i)C_r \right) \cdot P(mt + (i-1)\mu/k < T_f < mt + i\mu/k) \]

(3)

(2) The expected cost of an inspection renewal

1) The system is renewed by inspection at \( mt \) since the severe defective stage is found at \( mt \) before the minor defective stage is found, see Figure 3. The probability of such an inspection renewal is
\[ P(T_r = mt) \]
\[ = P((m-1)\mu < X_1 < mt, 0 < X_2 < mt - X_1, X_3 > mt - X_1 - X_2) \]
\[ = \int_{(mt-1)\mu}^{mt} f_{X_1}(x) \int_0^{mt-x} f_{X_2}(y)(1 - F_{X_1}(mt - x - y))dydx \]

\textbf{Figure 3: An Inspection Renewal at }\text{\( mt \) before the Minor Defective Stage is found}

2) The system is renewed at inspection \( mt + i\mu/k \) after the minor defective stage is firstly found at \( mt \) and the severe defective stage at \( mt + i\mu/k \), see Figure 4. The probability of such an inspection renewal is given by
\[ P(T_r = mt + i\mu/k) \]
\[ = P((m-1)\mu < X_1 < mt, mt + i\mu/k - X_1 < X_2 < mt + (i+1)\mu/k - X_1, \]
\[ \max X_3 > mt + i\mu/k - X_1 - X_2) \]
\[ = \int_{(mt-1)\mu}^{mt} f_{X_1}(x) \int_{mt+i\mu/k-x}^{mt+(i+1)\mu/k-x} f_{X_2}(y)(1 - F_{X_1}(mt + i\mu/k - x - y))dydx \]

The expected cost of inspection renewal can be derived in the similar way as Eq. (3).
\[ E(C_r(t)) = \sum_{m=1}^{\infty} \left( C_r + mC_r \right) \cdot P(T_r = mt) + \sum_{i=1}^{\infty} \left( C_r + (m+i)C_r \right) \cdot P(T_r = mt + i\mu/k) \]

(6)
3.2 The Expected Renewal Cycle Length

1) The expected length of a failure renewal

The pdf of failure at $T_f \in ((m-1)t,(m-1)t+z), z \in (0,t)$ is given as Eq. (4) in [6]. Accordingly, the expected renewal cycle length of a failure renewal is formulated as

$$E(L(t)) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \int_{z}^{z+dz} \left( \int_{T_f}^{(m-1)t+z} P(T_f = (m-1)t + z) \right) dz$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \int_{z}^{z+dz} (mt + it/k + z) \cdot P(T_f = (m-1)t + z) dz$$

(7)

2) The expected length of an inspection renewal

The expected length of an inspection renewal is given as

$$E(L_i(t)) = \sum_{i=0}^{\infty} [mt - P(T_f = mt) + \sum_{j=0}^{\infty} (mt + it/k) \cdot P(T_f = mt + it/k)]$$

(8)

3.3 The Expected Cost Per Unit Time

The purpose of the proposed model is to search the optimal inspection interval and shortening proportion of the inspection interval once the minor defective stage is detected at inspection by minimizing the expected cost per unit time. Based on the expected renewal cycle cost and length with different renewals scenarios, the expected cost per unit time is given by [7]

$$C(t) = \frac{E(C_f(t)) + E(C_p(t))}{E(L_f(t)) + E(L_i(t))}$$

(9)

4. Numerical Example

A numerical example is presented to show the applicability of the proposed model by minimizing the expected cost per unit time in this section. Here, we assume these three stages follow three Weibull distributions and the pdf of the $n$th ($n=1, 2$ and $3$) stage is

$$f_{X_n}(x) = \frac{b_n}{a_n} \left( \frac{x}{a_n} \right)^{b_n-1} e^{-\left( \frac{x}{a_n} \right)^b}$$

(10)

where $a_n > 0$ is the scale parameter and $b_n > 0$ is the shape parameter of the distribution.

Table 1: The Distribution Parameters

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$b_1$</td>
<td>$a_2$</td>
<td>$b_2$</td>
<td>$a_3$</td>
<td>$b_3$</td>
</tr>
<tr>
<td>43.478</td>
<td>1.15</td>
<td>17.241</td>
<td>1.47</td>
<td>6.25</td>
<td>1.14</td>
</tr>
</tbody>
</table>

The distribution parameters are shown in Table 1. The cost parameters are $C_f = 100$, $C_p = 1000$, $C_f = 5000$ respectively. According to the given parameters, the result of the proposed model is shown in Figure 5 with $k$ from 2 to 6 and $t$ from 5 to 20. The optimal inspection interval $t^* = 13$ and if the minor defective stage is identified by inspection, the optimal shortening proportion of the inspection interval $k^* = 5$. The optimal value of the expected cost per unit time $C^*(13) = 44.7457$. It is obvious that the inspection policy of the numerical example is different from that of the work by Wang [7] where shortening the inspection interval to be half of the current interval is optimal according to the given
parameters. This means that once the minor defective stage is identified at an inspection, the optimal measure is to shorten the inspection interval for the subsequent inspections. However, the shortening proportion may be different depending on the given distribution and cost parameters.

![Inspection interval vs. shortening proportion and cost](image)

**Figure 5:** Output Results of the Proposed Model

### 5. Conclusions

In order to investigate the optimal shortening proportion of the inspection interval after the minor defective stage is found by inspection, an inspection optimization model based on the three-stage failure process is proposed in this paper. The objective of the model is to search the optimal inspection interval and the optimal shortening proportion of the inspection interval by minimizing the expected cost per unit time. The results obtained from the numerical example show that shortening the inspection interval is optimal but the shortening proportion is different according to different parameters.

### References


**Rui Feng Yang** is a Master student of Mechanical Engineering College, Shijiazhuang, China.

**Fei Zhao** is a Ph.D student of Dongling School of Economics and Management, University of Science and Technology Beijing, China.

**Jianshe Kang** is currently professor of Mechanical Engineering College, Shijiazhuang, China.

**Xinghui Zhang** is a Ph.D. student of Mechanical Engineering College, Shijiazhuang, China.