Accelerated Test Planning with Independent Competing Risks and Concave Degradation Path

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Abstract: A step-stress test in the presence of competing risks and using degradation measurements is considered. It is assumed that underlying degradation path follows a concave degradation model and the intensity functions corresponding to competing risks depend only on the level of degradation. In this work, information from step-stress test at high level of stress is extrapolated, through a tempered failure rate model, to obtain the estimates of intensity functions at normal use conditions. No assumptions are made about failure times distribution. Finally, the results are used to estimate reliability function.

Keywords: Accelerated tests, competing risks, degradation process, intensity function, tempered failure rate model.

1. Introduction

Today, many important products are designed to operate without any failures for several years. Thus, very few units will fail in a test over a particular length of time at normal conditions of use. Hence, we need to use accelerated life tests to obtain failure information more quickly. Accelerated test planning with a single cause of failure, assuming log-location-scale lifetime distribution has already been studied in literature.

Recently, the planning problem of accelerated life test with multiple failure modes has been developed by some authors. For example, modeling and planning of step-stress accelerated life tests with independent competing risks are considered by Liu and Qiu (2011). Liu and Tang (2010) developed the planning problem of accelerated life tests for repairable systems in the presence of multiple independent causes and Pascal (2007)-(2010) studied the designs of accelerated life tests under independent Weibull and Lognormal causes of failure.

In this work, we assumed that a test unit may fail due to one of several causes, called competing risks. Further, we assumed that the intensity functions of competing risks depend only on degradation level of unit. In some reliability studies, it is possible to measure degradation level of unit at the moment of failure. Such degradation data can provide more reliability information than would be otherwise available only from accelerated failure time data. We used such degradation data in our plan and constructed a joint model which reflects the relationship between failure time and degradation data. Simultaneous analysis of degradation and failure time has been studied by Bagdonavičius et al. (2004), (2005) and Haghighi (2005).

This work aims to develop the previous studies under accelerated life tests and provide a design of step-stress accelerated test with independent causes of failure where the degradation of unit at the moment of failure would be available. Then information from test at high level of stress is extrapolated, through a tempered failure rate model, to obtain the estimates of intensity functions at normal use conditions. Our motivations for considering this work are as follows. The first motivation is based on the assumption on
the lifetime distribution of failure times. Our proposed plan does not have the limitations of considering a specific distribution for failure times. The second motivation is based on the using of degradation data that can provide more information. The rest of this paper is organized as follows. Section 2 presents a statistical formulation of joint model. Section 3 provides the maximum likelihood estimates of unknown parameters. An example is given to evaluate the performance of the method in Section 4. Finally, Section 5, presents a conclusion.

**Notation:**

- $A$: random vector related to each unit
- $g(t,a)$: concave degradation model
- $h(z,a)$: inverse function of $g$
- $h'$: first derivative of $h$
- $l$: index of stress level
- $n$: total number of test units
- $n_l$: number of failure at $l$th level of stress
- $S_0$: normal stress
- $S_1$: accelerated stress
- $S_{l}^{(k)}(t|a)$: conditional survival function under $l$th level of stress corresponding to $k$th competing risk
- $S(t), \hat{S}(t)$: survival and estimated survival functions
- $s$: number of competing risks
- $\eta, \beta$: parameters of Weibull distribution
- $\alpha^{(k)}$: factor corresponding to $k$th competing risk
- $\tau$: changing-point of stress
- $\lambda^{(k)}$: intensity function corresponding to $k$th competing risk
- $\pi, \pi'$: distribution function and pdf of $A$
- $\gamma_k$: vector of unknown parameters related to $k$th competing risk
- $\delta_i$: observed failure mode of $i$th unit
- $T_l$: failure time at $l$th level of stress
- $T_l^{(k)}$: failure time at $l$th level of stress due to $k$th competing risk
- $L_l, L$: likelihood function under $l$th level of stress, likelihood function
- $\Delta$: indicator of failure
- $Z(t)$: degradation process
2. The Model

We have made some assumptions as follows:
1) Two stress levels $S_0$ and $S_1$ ($S_0 < S_1$) are used. $S_0$ is stress at normal use conditions.
2) Intensity function corresponding to $k$th competing risk $\lambda^{(k)}$ at each level of stress depends only on degradation level of unit and as a rule it is an increasing function.
3) A concave degradation path model and a tempered failure rate model (Bhattacharyya and Soejoeti 1989) are hold.

The test is conducted as follows. All test units are initially placed on normal stress $S_0$, and run until time $\tau$ (changing-point of stress). Then, the stress is changed to high stress $S_1$, and the test continues until all remaining units fail.

Failure time of unit at $l$th level of stress $l=0, 1$ denotes by $T_l = \min (T^{(1)}_l, \ldots, T^{(s)}_l)$, where $T^{(k)}_l, k = 1, \ldots, s$ is the failure time corresponding to $k$th competing risk. Let $\Delta$ denotes the indicator of the failure due to competing risks:

$$\Delta = \begin{cases} 
1, & T_l = T^{(1)}_l, \\
2, & T_l = T^{(2)}_l, \\
\vdots & \vdots \\
s, & T_l = T^{(s)}_l.
\end{cases}$$

Consider at each level of stress, degradation process $Z$, is described by a concave degradation path model as follows:

$$Z(t) = \begin{cases} 
g(t, A_0), & t \leq \tau, \\
g(\tau, A_0) + g(t-\tau, A_1), & t > \tau.
\end{cases}$$

where, $g$ is a concave function such that $g(0, \cdot) = 0$ and $A_0$ and $A_1$ are random vectors with distribution functions $\pi_0$ and $\pi_1$ corresponding to each level of stress.

Figure 1 shows the simulated degradation path under constant and Step-Stress accelerated tests. Figure 1(a), shows a simulated concave degradation path under constant stress $S_0$ and $S_1$ whereas, Figure 1(b) shows the degradation path under simple step-stress test.
The relationship between accelerated degradation path and normal degradation path is shown in Figure 1(c). In this Figure, one can see the time $t$ under $S_1$ is equivalent with the time $t_\circ$ under $S_0$.

Figure 1(c): Simulated Degradation Path under Constant and Step-Stress Accelerated Tests

Let $n_1$ and $n_2$ be the number of failures at normal and accelerated conditions. For each unit, our approach collects the (accelerated) failure time, the (accelerated) degradation level at the moment of failure and the type of failure. Moreover, for the units which are not failed under first level of stress (normal conditions), the degradation level at the changing-point of stress is recorded $(z^*_{1}, \ldots, z^*_{n_2})$. This information is served to obtain the value of $A$ under first level of stress, denoted by $a_0$, for these units. Table 1 summarizes the observations at the end of test.

<table>
<thead>
<tr>
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<th>Under $S_0$</th>
<th>at $\tau$</th>
<th>Under $S_1$</th>
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<tbody>
<tr>
<td>$T$</td>
<td>$A$</td>
<td>$Z$</td>
<td>$T$</td>
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<tr>
<td>$t_1$</td>
<td>$\delta_1$</td>
<td>$z_1$</td>
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<td>$t_2$</td>
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<tr>
<td>$t_{n_1}$</td>
<td>$\delta_{n_1}$</td>
<td>$z_{n_1}$</td>
<td>$t_{n_2}$</td>
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Let $\lambda_i^{(k)}(z)$ and $S_i^{(k)}(A = a)$ are respectively, the intensity and conditional survival functions corresponding to $k$th risk at $l$th level of stress.

$$S_0^{(k)}(t \mid A = a) = P\left\{ T_0^{(k)} > t \mid A = a \right\}$$

$$= \exp\left\{ - \int_0^t \lambda_0^{(k)}(s,g,a_0)ds \right\}$$

(3)
\[ S^{(k)}_1(t \mid A = a) = S^{(k)}_0(\tau \mid A = a) P_{\gamma_{1}^{(k)}} \succ t \mid A = a] = S^{(k)}_0(\tau \mid A = a) \exp \left\{ - \int_{\tau}^{t} \lambda_{a_0}^{(k)}(s) g(s - \tau, a_i) ds \right\} \] (4)

where, \( a_0, a_1 \) are respectively, the value of \( A \) under first and second levels of stress.

Denote by \( h \) and \( h' \), the inverse function and partial derivation of \( g \), from (4) we have

\[ S^{(k)}_0(t \mid A = a) = \exp \left\{ - \int_{0}^{t} \lambda_{a_0}^{(k)}(w) h'(w, a_o) dw \right\} \] (5)

where, \( z_1 = g(t, a_o) \), and

\[ S^{(k)}_1(t \mid A = a) = \exp \left\{ - \int_{0}^{t} \lambda_{a_0}^{(k)}(u) h'(u, a_0, a) du \right\} \exp \left\{ - \int_{z_1}^{z_2} \lambda_{a_0}^{(k)}(u) h'(u - g(\tau, a_0), a_i) du \right\} \] (6)

where, \( z_1 = g(t, a_o) \) and \( z_2 = g(\tau, a_0) + g(t - \tau, a_1) \).

Under tempered failure rate model, we have

\[ \lambda^{(k)}(z) = \begin{cases} \lambda_{a_0}^{(k)}(z), & z < g(\tau, a_0), \\ \alpha^{(k)} \lambda_{a_0}^{(k)}(z), & z \geq g(\tau, a_0). \end{cases} \] (7)

The factor \( \alpha^{(k)} \), corresponding to \( k \)th competing risk depends on stress level. So,

\[ S^{(k)}_1(t \mid A = a) = \exp \left\{ - \int_{0}^{t} \lambda_{a_0}^{(k)}(u) h'(u, a_0, a) du \right\} \exp \left\{ - \int_{z_1}^{z_2} \alpha^{(k)} \lambda_{a_0}^{(k)}(u) h'(u - g(\tau, a_0), a_i) du \right\} \] (8)

From the assumption that the model is a tempered failure rate model, the conditional survival function of a test unit in the presence of competing risks and under simple step-stress test is given by

\[ S^{(k)}(t \mid A = a) = \begin{cases} S^{(k)}_0(t \mid A = a), & 0 \leq t < \tau, \\ S^{(k)}_1(t \mid A = a), & \tau \leq t < \infty. \end{cases} \] (9)

3. Estimation

We suppose that the intensities belong to a parametric class. Let \( \delta_i \) denote the type of failure. The likelihood function resulted from observed data under \( S_0 \) is:

\[ L' = \prod_{i=1}^{n} \lambda_{0}^{(\delta_i)}(\gamma_{\delta_i}, z_i) \prod_{k=1}^{K} S^{(k)}_0(\gamma_k, a_i, A_{0i}) \] (10)

where \( \gamma \) is a multi-dimensional parameter vector.

The likelihood function resulted from observations under \( S_1 \) is
Then we can write likelihood function as follows:

\[ L^2 = \prod_{i=1}^{n_2} \alpha_{j=1}^{(E)} \lambda_{0j}^{(E)} \left( \gamma_{jk} \bigg| z_i \right) \prod_{k=1}^{s} S_{1k}^{(k)} \left( \gamma_{k} \big| f_i, \alpha_{i} \right) \pi_{00}^{\prime} \left(A_{0} \right) \pi_{1}^{\prime} \left(A_{1} \right) \]

The log-likelihood function are obtained as:

\[ l \approx \sum_{i=1}^{n} \log \left[ \lambda_{0j}^{(E)} \left( \gamma_{jk} \bigg| z_i \right) \right] \]

\[ + \sum_{i=1}^{n} \sum_{k=1}^{s} \log \left[ S_{1k}^{(k)} \left( \gamma_{k} \big| f_i, \alpha_{i} \right) \right] \]

\[ + \sum_{i=1}^{n} \sum_{k=1}^{s} \log \left[ \pi_{00}^{\prime} \left(A_{0} \right) \pi_{1}^{\prime} \left(A_{1} \right) \right] \]

The MLE’s of \( \gamma_{k}, k = 1, \cdots, s \) are obtained by solving \( \frac{d l}{d \gamma_{k}} = 0. \)

The survival function under normal conditions is obtained as follows: (Bagdonavičius et al. 2005)

\[ S(t) = P\{ T_0 > t \} = P\{ \min(T_{01}, \ldots, T_{0s}) > t \} = \int_0^\infty \prod_{k=1}^s S_{0k}^{(k)} (t \big| A = a) d \pi(a) \]

and estimated survival function is obtained as:

\[ \hat{S}(t) = \int_0^\infty \prod_{k=1}^s \hat{S}_{0k}^{(k)} (t \big| A = a) d \pi(a) \]

where, \( \hat{S}_{0k}^{(k)} (t \big| A = a) = S_{0k}^{(k)} (t \big| a, \hat{\gamma}_k) \).

4. Simulation

We conducted a simulation study to evaluate the performance of the method. We considered \( g(t, A) = \left( \frac{t}{A} \right)^{0.5} \), where \( \pi(a) \), (distribution function of \( A \)) belongs to a Weibull family with parameters \( \left( \eta = 5, \beta_1 = 2.5 \right) \). It is assumed that the effect of stress changes only the scale parameter of \( \pi \). It means that

\[ \pi(a) = \left\{ \begin{array}{ll} W(\eta = 5, \beta_0 = 2.5), & \text{under } S_0, \\ W(\eta = 5, \beta_1 = 2.2), & \text{under } S_1. \end{array} \right. \]

Further, we assumed that two competing risks are possible and intensities belong to a parametric class as \( \lambda^z \left( z, \gamma_k \right) = \left( \frac{z}{\theta_k} \right)^{\nu_k} \), \( \gamma_k = \left( \theta_k, \nu_k \right), \ k = 1, 2. \)

This form of intensity has been observed in failures due to wearing, fatigue or mechanical damages.

For fixed values of \( (\nu_1, \nu_2, \theta_1, \theta_2, \alpha_1, \alpha_2) = (2, 4, 10, 15, 1.5, 2.2) \) and \( \tau = 15 \), using (5), (6) the failure times under \( S_0 \) and \( S_1 \) are obtained \( (n_1 = 25, n_2 = 15) \).

The maximum likelihood estimates are resulted by maximizing (8).
as \((\hat{\nu}_1, \hat{\nu}_2, \hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}_1, \hat{\alpha}_2) = (1.88, 4.2, 13.5, 16.8, 1.75, 2.82)\) and the estimated survival function is obtained by substituting the MLE’s in (10) as follows:

\[
\hat{S}(t) = \exp \left\{ - \sum_{k=1}^{3} \left( \frac{2t^{\tilde{\nu}_k^2}}{2} \right) \exp \left\{ - \left( \frac{a_0}{\tilde{\beta}_0} \right)^{\tilde{\nu}_k} \right\} \right\}
\]

Using Monte Carlo method, we approximated \(\hat{S}(t)\) and the result is given in Figure 2.

![Estimated Survival Function from Simulated Data](image)

**Figure 2**: Estimated Survival Function from Simulated Data

5. Conclusion

This paper deals with the accelerated test planning for the units with a concave degradation path. Multiple failure modes are considered. Instead of considering a specific distribution function for failure times, degradation data are used for modeling. However, we do require the assumption that the degradation path is known. In this approach, intensities under accelerated stress are related to intensities at normal conditions by tempered failure rate model. We derived the maximum likelihood estimates of the intensities and estimated the survival function at normal condition.

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References


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