Optimum Test Plan for 3-Step, Step-Stress Accelerated Life Tests

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Abstract: In this paper we obtain the optimum time for changing the stress level in 3-step, step stress accelerated life testing under the cumulative exposure model. The lifetimes of test units are assumed to follow a two-parameter Lomax distribution. The scale parameter of the Lomax failure time distribution at constant stress level is assumed to be a log-linear function of the stress level. We also assume that there may exist a quadratic relationship between the log mean failure time and stress. As an extension of the results for the linear model, the optimum plan for a quadratic model is also proposed. We derive the optimum test plans by minimizing the asymptotic variance of the maximum likelihood estimator of the mean life at a design (use) stress.

Keywords: Accelerated life testing, Lomax distribution, asymptotic variance, maximum likelihood estimate, cumulative exposure model.

1. Introduction

Today with highly reliable products, it is very difficult to obtain a reasonable amount of failure data under normal use condition. The accelerated life testing (ALTs) are used to overcome this problem. ALTs are done on greater stresses (such as, temperature, vibration, voltage, pressure, humidity, cycling rate etc.) than use stress and then ALTs quickly yield information on test units. Subsequently these are analyzed by selecting an appropriate physical model and then extrapolated to the design stress to estimate the lifetime distribution or parameter. Such over stress testing reduces time and cost of the test. In the literature, there exist currently three types of ALTs: constant-stress ALT, progressive-stress ALT and step-stress ALT.

Constant-stress ALT: The stress is kept at a constant level throughout the life of test product (e.g., Kielpinski and Nelson [1], Nelson and Meeker [2] and Meeker [3]).

Progressive-stress ALT: The stress applied to a test product is continuously increasing in time (e.g., Wang and Fei [4] and Abdel-Hamid and AL-Hussaini [5]).

Step-stress ALT: The stress is applied to test units in such a way that it will change at pre-specified times. Generally, a test unit starts at a specified low stress. If the unit does not fail at a specified time, stress on it is raised and held for a specified time. Further, stress is repeatedly increased until the test unit fails or the censoring time is reached. Simple step-stress tests use only two stresses in a test.

The step-stress scheme, with the cumulative exposure model was first introduced by Nelson in [6], defined for k-stress levels as follows:
\[ F(t) = \begin{cases} 
F_1(t), & 0 \leq t < \tau_1, \\
F_2(t-\tau_1+s_1), & \tau_1 \leq t < \tau_2, \\
F_3(t-\tau_2+s_2), & \tau_2 \leq t < \tau_3, \\
\vdots & \\
F_k(t-\tau_{k-1}+s_{k-1}), & \tau_k \leq t < \infty, 
\end{cases} \]  

(1)

where \( F_i(t) \) is the cumulative distribution function of the failure time at the \( i^{th} \) stress level, \( \tau_i \) is the time of change from \( i^{th} \) to \( (i+1)^{th} \) stress level and \( s_{i-1} \) is an equivalent start time at \( i^{th} \) stress level which would have produced the same population cumulative fraction failing. Thus, \( s_{i-1} \) is the solution of

\[ F_{i+1}(s) = F_i(\tau_i - \tau_{i-1} + s_{i-1}). \]

In the past few years, the work in step-stress ALT planning problems was commonly used to design an optimal plan by minimizing the asymptotic variance of the maximum likelihood estimate (MLE) of the logarithm of mean life or some percentile of life at a specified stress level. In [7] and [8], Miller et al. and Bai et al. presented a simple step-stress accelerated life test (SSALT) plan assuming that the failure lifetimes of units follow an exponential life distribution with censoring. Yeo and Tang derived a simple SSALT for an optimum hold time under low stress and an optimum low stress level by taking the target acceleration factor into consideration in [9] and [10]. Khamis and Higgins [11] considered quadratic stress-life relationship and derived the optimum three-step SSALT plan for the exponential distribution. They also introduced a compromise test plan, which is an alternative of optimum linear and quadratic test plans. Khamis and Higgins extended the 2-step stress work for m-step stress ALT with k stress variables, assuming complete knowledge of the stress-life relation with multiple stress variables in [12]. These studies were based on the assumption that the failure time follows an exponential distribution because of its simplicity.


The Lomax distribution was originally proposed as a second kind of the Pareto distribution by Lomax[20]. It is a useful model in a wide variety of socioeconomic as well as lifetime contexts. It has been used in the analysis of income data, and business failure data. Also, it has been used to provide a good model in biomedical problems. It may describe the lifetime of a decreasing failure rate component and has been recommended by Bryson in [21] as a heavy tailed alternative to the exponential distribution. In reliability theory it appears as a mixture of the one parameter Exponential distribution.
The probability density function (p.d.f.) of a random variable that has the Lomax distribution is given by:

\[ f(t, \theta, \lambda) = \lambda \theta^{-1} \left(1 + \frac{t}{\theta}\right)^{-1(1+\lambda)}, \ t > 0, \ \lambda > 0, \ \theta > 0 \]  

(2)

In this paper, we have developed an optimum plan for 3-step, step-stress ALT, which assumes that a linear as well as quadratic relationship exists between the log mean failure time and the stress. It is derived by minimizing the asymptotic variance of the MLE of the model parameters at a design stress with respect to stress change time.

The proposed model and assumptions are given in section 2. The MLE and the Fisher information matrix are derived for the parameter estimation in section 3. The optimum plans under linear as well as quadratic models are discussed in section 4. Section 5 contains the confidence interval for the model parameter, and the issue of hypothesis testing is described. The numerical study is performed with an example in section 6 and the conclusion of the proposed study is summarized in section 7.

**Notation:**

- \( X_0, X_1, X_2, X_3 \): stresses level (design, low, medium and high)
- \( \xi \): extrapolation amount where \( \xi = (X_1 - X_0)/(X_2 - X_1) \)
- \( n \): number of test units
- \( n_i \): number of failed units at stress \( X_i, i=0,1,2,3 \)
- \( t_{ij} \): failure time of test unit j at stress \( X_i, i=1,2,3, j=1,2, \ldots n_i \)
- \( \theta_i \): mean life at \( X_i, i=0,1,2,3 \)
- \( \tau \): stress changing point
- \( \tau^* \): optimum time of changing stress
- \( F_i(t) \): c.d.f. of Lomax distribution with mean \( \theta_i \), \( i=1,2,3 \)
- \( F(t) \): c.d.f. of a test unit under step-stress
- \( \lambda \): shape parameter
- \( \beta_0, \beta_1 \): parameters of the log-linear relationship between stress \( X_i \) and mean life \( \theta_i \) under Linear Model
- \( \beta_0', \beta_1', \beta_2' \): parameters of the log-linear relationship between stress \( X_i \) and mean life \( \theta_1 \) under Quadratic Model
- \( \text{MLE} \): maximum likelihood estimate
- \( \text{AVarL} \): asymptotic variance for the Linear Model.
- \( \text{AVarQ} \): asymptotic variance for the Quadratic Model

**2. Step-Stress Model and Assumptions**

Under any constant stress, the time to failure of a test unit follows a Lomax distribution with distribution function:

\[ F(t, \theta, \lambda) = 1 - (1 + \frac{t}{\theta})^{-\lambda}, \ t > 0, \ \lambda > 0, \ \theta > 0 \]  

(3)

From equation (1), the cumulative exposure model for a 3-step step-stress is given by:

\[ F(t) = \begin{cases} 
F_1(t), & 0 \leq t < \tau_1 \\
F_2(t - \tau_1 + s_1), & \tau_1 \leq t < \tau_2 \\
F_3(t - \tau_2 + s_2), & \tau_2 \leq t < \infty
\end{cases} \]  

(4)

with \( s_1 \) the solution of \( F_3(s) = F_1(\tau_1) \) and \( s_2 \) the solution of \( F_3(s_2) = F_2(\tau_2 - \tau_1 + s_1) \). Thus, the Lomax cumulative exposure model for a 3-step step-stress is given by
F(t) = \begin{cases} 
1 - \left[1 + \frac{t}{\theta_1}\right]^{-\lambda}, & 0 \leq t < \tau_1 \\
1 - \left[1 + \frac{t - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}\right]^{-\lambda}, & \tau_1 \leq t < \tau_2 \\
1 - \left[1 + \frac{t - \tau_2}{\theta_3} + \frac{\tau_1 - \tau_2}{\theta_2} + \frac{\tau_1}{\theta_1}\right]^{-\lambda}, & \tau_2 \leq t < \infty 
\end{cases} 
(5)

From equation (4), \( s_1 = \tau_1 (\theta_2/\theta_1) \) and \( s_2 = \left(\tau_2 - \tau_1 + \frac{\tau_1 (\theta_2)}{\theta_1}\right) \left(\frac{\theta_2}{\theta_1}\right) \). The corresponding p.d.f. of the failure time is obtained as follows:

\[
f(t) = \begin{cases} 
\frac{\lambda}{\theta_1} \left[1 + \frac{t}{\theta_1}\right]^{\lambda-1}, & 0 \leq t < \tau_1 \\
\frac{\lambda}{\theta_2} \left[1 + \frac{t - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}\right]^{\lambda-1}, & \tau_1 \leq t < \tau_2 \\
\frac{\lambda}{\theta_3} \left[1 + \frac{t - \tau_2}{\theta_3} + \frac{\tau_1 - \tau_2}{\theta_2} + \frac{\tau_1}{\theta_1}\right]^{\lambda-1}, & \tau_2 \leq t < \infty 
\end{cases} 
(6)
\]

The following assumptions are made:

1. Testing is done at stresses \( X_1, X_2 \) and \( X_3 \), where \( X_1 < X_2 < X_3 \).
2. The scale parameter \( \theta_i \) at stress level \( i, i=0,1,2,3 \) is a log-linear function of stress, i.e., \( \log(\theta_i) = \beta_0 + \beta_1 X_i \) (7)
   
   where \( \beta_0 \) and \( \beta_1 \) are unknown parameters depending on the nature of the product and method of the test.
3. The lifetimes of test units are independent and identically distributed.
4. The constant \( \lambda \): independent of time and stress.
5. All \( n \) units are initially placed on low stress \( X_1 \) and run until pre-specified time \( \tau_1 \) when the stress is changed to high stress \( X_2 \) for those remaining units that have not failed. The test is continued until pre-specified time \( \tau_2 \) when stress is changed to \( X_3 \), and continued until all remaining units fail.

### 3. Maximum Likelihood Estimation and Fisher Information Matrix

In order to obtain the MLE of the model parameters, let \( t_{ij}, i=1,2,3, j=1,2,3, \ldots, n_i \) be the observed failure time of a test unit \( j \) under the stress level \( i \), where \( n_i \) denotes the number of units failed at the stress level \( i \). The likelihood function is given by

\[
L(t; \theta_1, \theta_2, \theta_3, \lambda) = \prod_{i=1}^{n_1} \left[\frac{\lambda}{\theta_1} \left[1 + \frac{t_{1i}}{\theta_1}\right]^{\lambda-1}\right] \cdot \prod_{j=1}^{n_2} \left[\frac{\lambda}{\theta_2} \left[1 + \frac{t_{2j} - \tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}\right]^{\lambda-1}\right] \cdot \prod_{i=1}^{n_3} \left[\frac{\lambda}{\theta_3} \left[1 + \frac{t_{3i} - \tau_2 + \tau_2 - \tau_1 + \frac{\tau_1}{\theta_1}}{\theta_2} + \frac{\tau_1}{\theta_1}\right]^{\lambda-1}\right] 
(8)
\]

Taking the logarithm of the likelihood function and letting \( n = n_1 + n_2 + n_3 \), we get

\[
\log L(t; \theta_1, \theta_2, \theta_3, \lambda) = n \log \lambda - n_1 \log \theta_1 - n_2 \log \theta_2 - n_3 \log \theta_3
\]
Solving the equation (13) for \( \lambda \) yields

\[
\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{j=1}^{n_1} \log \left[ 1 + \frac{t_{ij}}{\theta_1} \right] - \sum_{j=1}^{n_2} \log \left[ 1 + \frac{t_{ij} - t_1}{\theta_2} + \frac{t_2 - t_1}{\theta_2} + \frac{t_3}{\theta_1} \right] \\
+ \sum_{j=1}^{n_3} \log \left[ 1 + \frac{t_{3j} - t_2}{\theta_3} + \frac{t_2 - t_1}{\theta_2} + \frac{t_3}{\theta_1} \right] = 0
\] 

Solving the equation (13) for \( \lambda \) yields,
where
\[A = \sum_{j=1}^{n_1} \log[1 + (t_{1j} - \tau_1\theta_1^{-1})] + \sum_{j=1}^{n_2} \log[1 + (t_{2j} - \tau_2\theta_2^{-1}) + (\tau_2 - \tau_1\theta_1^{-1})] \]
\[+ \sum_{j=1}^{n_3} \log[1 + (t_{3j} - \tau_2\theta_3^{-1}) + (\tau_2 - \tau_1\theta_1^{-1})] \]

By substituting the value of \( \lambda \) given by (14) in (11) and (12) and equating to zero, we get the two nonlinear equations given in Appendix (A.1). Obviously, it is not possible to obtain a closed-form solution for these equations. The iterative Newton-Raphson algorithm is thus used to approximate the MLE of \( \beta_0 \) and \( \beta_1 \). Once estimates of \( \beta_0 \) and \( \beta_1 \) have been determined, an estimate of \( \lambda \) can be obtained from the equation (14).

Under the assumption 4, the Fisher information matrix for \( n \) samples is obtained by taking the mean of the second and mixed partial derivatives of the log likelihood function (10) with respect to \( \beta_0 \) and \( \beta_1 \). The elements of above defined matrix are given as follows:
\[F_{11} = \frac{\lambda}{\lambda + 2} \]
\[F_{12} = F_{21} = \frac{\lambda}{\lambda + 2} [X_1 + (X_2 - X_1)I_1 + (X_3 - X_2)I_2] \]
\[F_{22} = \frac{\lambda}{\lambda + 2} [X_1^2 + (X_2^2 - X_1^2)I_1 + (X_3^2 - X_2^2)I_2] \]

where
\[I_1 = \left[1 + \frac{\xi}{\eta_{\beta_0}}\right]^{(\beta_0 + 1)} \]
\[I_2 = \left[1 + \frac{\xi^2 - \eta_{\beta_0} \xi}{\eta_{\beta_0}^2} + \frac{\xi^2 - \eta_{\beta_0} \xi}{\eta_{\beta_0}^2}\right]^{(\beta_0 + 1)} \]

Thus, the Fisher information matrix can be written as:
\[F = n \frac{\lambda}{\lambda + 2} \left[ \begin{array}{cc} X_1 + (X_2 - X_1)I_1 + (X_3 - X_2)I_2 & X_1^2 + (X_2^2 - X_1^2)I_1 + (X_3^2 - X_2^2)I_2 \\ \end{array} \right] \]

The asymptotic variance of the log mean life at normal (design) use stress \( X_0 \) is given by:
\[n\text{VarLog}(\theta_0) = \text{VarLog}(\hat{\beta}_0 + \hat{\beta}_1 X_0) \]
\[= \left( \frac{\lambda + 2}{\lambda} \right) \left[ \frac{\xi^2 + \eta_{\beta_0} \xi I_1 + 2\eta_{\beta_0} \xi I_1 + (1 - \eta_{\beta_0}^2 + 2\xi - 2\eta_{\beta_0}^2) I_2}{\eta_{\beta_0}^2 (I_1 - I_2) + I_2 - (\eta_{\beta_0} I_1 - I_2 + I_2)^2} \right] \]

where \( \xi = \frac{X_1 - X_0}{X_3 - X_1} \) and \( \eta = \frac{X_2 - X_0}{X_3 - X_1} \) and \( \hat{\cdot} \) stands for the MLE.

The optimum linear plan can be obtained by minimizing the asymptotic variance of the log mean life estimate in (17) at normal (design) stress when \( \tau_1 = \tau_2 \), so that only two extreme stresses \( X_1 \) and \( X_3 \) are used in testing. Hence, the optimum stress changing time is
\( \tau^* = \theta_1 \left[ \left( \frac{1+2\zeta}{\zeta} \right)^{\frac{1}{\lambda+1}} - 1 \right] \) \tag{18}

The derived optimum linear plan uses only two extreme stresses \( X_1 \) and \( X_3 \) without \( X_2 \), which is infeasible. Since three stresses \( X_1, X_2 \) and \( X_3 \) are required for optimum plan under 3-step, step-stress ALT experiment, therefore, we prefer the compound linear plan to obtain the stress changing times given in section 4.

4. Proposed Optimum Plans

4.1 Compound Linear Plan:

According to Khamis and Higgins [11], we applied the compound linear, preceded as follows:

We fix stress levels and choose \( \tau_1 \) and \( \tau_2 \) at which to change stresses. Our compromise plan uses the optimum simple step stress plan twice.

- Choose \( \tau_1 \) to be the optimum value for changing stress in the step-stress model with stresses \( X_0, X_1, X_2 \).
- Let \( \tau_1' \) be the optimum time for changing stress in the simple step-stress model with stresses \( X_0, X_2, X_3 \).
- Let \( \tau_2 = \tau_1 + \tau_1' \).

4.2 Quadratic Plan:

If the sample ALT data show a curvilinear relationship between the log mean life \( \theta_i' \) and stress \( X_i \), it might be appropriate to fit the quadratic model:

\[
\log(\theta_i') = \beta'_0 + \beta'_1 X_i + \beta'_2 X_i^2, \quad i = 0, 1, 2, 3. \tag{19}
\]

For this quadratic model, the testing can be done using a 3-step, step-stress ALT. Thus, the log likelihood function (10) can be extended to the quadratic model after substituting the value of \( \theta_i' \) from equation (19), given by:

\[
\log L(t; \beta'_0, \beta'_1, \beta'_2, \lambda') = n \log \lambda' - n_1(\beta'_0 + \beta'_1 X_1 + \beta'_2 X_1^2) - n_2(\beta'_0 + \beta'_1 X_2 + \beta'_2 X_2^2)
\]

\[
- n_3(\beta'_0 + \beta'_1 X_3 + \beta'_2 X_3^2) - (\lambda' + 1) \left( \sum_{j=1}^{n_1} \log \left[ 1 + \tau_1 e^{-\left( \beta'_0 + \beta'_1 X_1 + \beta'_2 X_1^2 \right)} \right] \right)
\]

\[
+ \sum_{j=1}^{n_2} \log \left[ 1 + (\tau_{2j} - \tau_1) e^{-\left( \beta'_0 + \beta'_1 X_2 + \beta'_2 X_2^2 \right)} + \tau_1 e^{-\left( \beta'_0 + \beta'_1 X_1 + \beta'_2 X_1^2 \right)} \right]
\]

\[
+ \sum_{j=1}^{n_3} \log \left[ 1 + (\tau_{3j} - \tau_2) e^{-\left( \beta'_0 + \beta'_1 X_3 + \beta'_2 X_3^2 \right)} + (\tau_2 - \tau_1) e^{-\left( \beta'_0 + \beta'_1 X_2 + \beta'_2 X_2^2 \right)} \right]
\]

\[
+ \tau_2 e^{-\left( \beta'_0 + \beta'_1 X_1 + \beta'_2 X_1^2 \right)} \right) \right) \] \tag{20}

The MLEs, \( \beta'_0, \beta'_1, \beta'_2 \) and \( \lambda' \) of the model parameters \( \beta'_0, \beta'_1, \beta'_2 \) and \( \lambda' \) are the values which maximize the likelihood function, which is obtained by taking the first derivative of the logarithm of the likelihood functions (20) with respect to \( \beta'_0, \beta'_1, \beta'_2 \) and \( \lambda' \) and equating to zero:
\[
\frac{\partial \log L}{\partial \beta_2} = -n_1 X_1^2 - n_2 X_2^2 - n_3 X_3^2 \\
+ (\lambda' + 1) \left\{ \sum_{j=1}^{n_1} X_1^2 t_{j1} \theta_1^{-1} + \sum_{j=1}^{n_2} X_2^2 (t_{j2} - \tau_1) \theta_2^{-1} + X_3^2 t_{j3} \theta_3^{-1} \right\} \\
+ \sum_{j=1}^{n_3} X_3^2 (t_{j3} - \tau_2) \theta_3^{-1} + X_3^2 (t_{j2} - \tau_2) \theta_2^{-1} + X_3^2 t_{j3} \theta_3^{-1} \right\} = 0
\]  

(21)

The other maximum likelihood equations of \( \beta_0, \beta_1 \) and \( \lambda \) are provided in a supplementary document available upon request from the corresponding author. The asymptotic Fisher information matrix is given by:

\[
F_1 = \begin{bmatrix}
-\frac{\partial^2 \log L}{\partial \beta_0^2} & \frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} & \frac{\partial^2 \log L}{\partial \beta_2^2} \\
-\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_1} & -\frac{\partial^2 \log L}{\partial \beta_1^2} & \frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2} \\
-\frac{\partial^2 \log L}{\partial \beta_0 \partial \beta_2} & \frac{\partial^2 \log L}{\partial \beta_1 \partial \beta_2} & -\frac{\partial^2 \log L}{\partial \beta_2^2}
\end{bmatrix}
\]  

(22)

where the elements of \( F_1 \) are given in the aforementioned supplementary document. Then,

\[
\text{AVarQlog} \left( \hat{\beta}_0 \right) = \text{AVarQ} \left( \hat{\beta}_0 + \hat{\beta}_1 X_0 + \hat{\beta}_2 X_3^2 \right) = HF_1^{-1}H^T
\]  

(23)

where \( F_1 \) is the asymptotic Fisher information matrix given in equation (22) and

\[
H = \begin{bmatrix}
\frac{\partial \log \theta_0}{\partial \beta_0} & \frac{\partial \log \theta_0}{\partial \beta_1} & \frac{\partial \log \theta_0}{\partial \beta_2}
\end{bmatrix}
\]

Then,

\[
\text{AVarQlog} \left( \hat{\theta}_0 \right) = \left[ 1 \ X_0 X_3^2 \right] F_1^{-1} \left[ 1 \ X_0 X_3^2 \right]^T
\]  

(24)

where, \( T \) stands for transpose. Therefore, the optimum stress change times \( \tau_1^* \) and \( \tau_2^* \), minimizing the AVarQ is the log mean life at the design stress given in (23).

5. Confidence Interval and Hypothesis Testing

Confidence intervals for the population parameters \( \beta_0 \) and \( \beta_1 \) are given as:

\[
L_{\beta_1} = \hat{\beta}_1 - z \sqrt{\text{Var}(\hat{\beta}_1)}, \quad U_{\beta_1} = \hat{\beta}_1 + z \sqrt{\text{Var}(\hat{\beta}_1)}
\]

\[
L_{\beta_0} = \hat{\beta}_0 - z \sqrt{\text{Var}(\hat{\beta}_0)}, \quad U_{\beta_0} = \hat{\beta}_0 + z \sqrt{\text{Var}(\hat{\beta}_0)}
\]  

(25)

Test of hypothesis for these parameters can be performed by using either the likelihood ratio test or the approximate normality of the MLEs in large sample sizes. In the latter case, it is most convenient to use approximation.
\[ \begin{align*} 
(\hat{\beta}_0, \hat{\beta}_1) \sim N \left( (\beta_0, \beta_1), F^{-1} \right). 
\end{align*} \]

An important inference problem concerning the regression coefficient \((\beta_0, \beta_1)\) is the test of hypothesis \(H_0': \beta_0 = 0\) against \(H_1': \beta_0 \neq 0\).

The test with approximate size \(\gamma\) is given by the following decision rule: reject \(H_0\) if and only if \(\Lambda > \chi_{1-\gamma,1}^2\), where \(\chi_{1-\gamma,1}^2\) is the \((1-\gamma)\)th quantile of the Chi-square distribution with one degree of freedom, where

\[ \Lambda = -2\log \frac{L(\hat{\beta}_0, 0)}{L(\hat{\beta}_0, \hat{\beta}_1)} \]  

(26)

6. Numerical Study
For the numerical investigation, the R code developed is available from the corresponding author.

Example 1: We simulated 50 random observations with initial values \(X_0=0.1, X_1=0.3, X_2=0.5, X_3=1.0, \beta_0=1, \beta_1=2\) and \(\lambda=1\). Both the changing stress time are obtained as: \(\tau_1=3.62\) and \(\tau_2=9.55\).

<table>
<thead>
<tr>
<th>Stress levels</th>
<th>Failure Times</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X_1=0.3)</td>
<td>0.110 0.218 0.318 0.545 0.592 0.732 1.051 1.234 1.289 1.778 2.092 2.521 2.816</td>
</tr>
<tr>
<td>(X_2=0.5)</td>
<td>4.100 4.465 4.748 5.126 6.704 8.092 8.588</td>
</tr>
</tbody>
</table>

- From the simulated data, we fit the following model

\[ \log(\theta_i) = \beta_0 + \beta_1 X_i, \quad i = 1, 2, 3 \]

The MLE of \(\beta_0\) and \(\beta_1\) are obtained using R software. The MLE are

\[ \hat{\beta}_0 = 3.373907 \quad \text{and} \quad \hat{\beta}_1 = -1.033235 \]

- The Fisher information matrix is obtained as

\[ F^{-1} = \begin{bmatrix} 0.1541332 & -0.1639948 \\ -0.1639948 & 0.2578626 \end{bmatrix} \]

Table 2: Numerical Values of Stress Changing Times with Variation in Scale and Shape

<table>
<thead>
<tr>
<th>(\theta_i)</th>
<th>(\theta_2)</th>
<th>(\lambda)</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
<th>(\tau_1)</th>
<th>(\tau_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>2.5</td>
<td>3.5</td>
<td>4.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 3</td>
<td>3.45 7.03</td>
<td>1.65 3.46</td>
<td>1.07 2.27</td>
<td>0.79 1.69</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 5</td>
<td>6.90 12.87</td>
<td>3.30 6.31</td>
<td>2.15 4.15</td>
<td>1.59 3.08</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 7</td>
<td>10.35 18.71</td>
<td>4.95 9.17</td>
<td>3.22 6.02</td>
<td>2.38 4.48</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8 9</td>
<td>13.81 24.56</td>
<td>6.60 12.02</td>
<td>4.29 7.90</td>
<td>3.18 5.87</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 3: Hypothesis is Tested for Parameters in the Model \( \log(\theta_i) = \beta_0 + \beta_1 X_i \)

<table>
<thead>
<tr>
<th>Model</th>
<th>LogL</th>
<th>-2logΛ</th>
<th>d.f.</th>
<th>( \chi^2(0.05) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full model(( \beta_0, \beta_1 ))</td>
<td>-184.4223</td>
<td>110.3136</td>
<td>1</td>
<td>3.84</td>
</tr>
<tr>
<td>( \beta_0=0 )</td>
<td>-239.5791</td>
<td>- &amp;  4.1786</td>
<td>1</td>
<td>3.84</td>
</tr>
<tr>
<td>( \beta_1=0 )</td>
<td>-186.5116</td>
<td>4.1786</td>
<td>1</td>
<td>3.84</td>
</tr>
</tbody>
</table>

The observed value of \( F^{-1} \), i.e., \( F^{-1} \), is obtained by substituting the estimated values of parameters \( \hat{\beta}_0 \) and \( \hat{\beta}_1 \) for the true parameters in the asymptotic Fisher information matrix.

The 95\% confidence limits for the estimate of the parameters are

\[
2.604415 \leq \beta_0 \leq 4.143399; \quad -2.028526 \leq \beta_1 \leq -0.0379432
\]

Table 3 shows the likelihood ratio statistic \( \Lambda \) for the test of various sub-models vs full model. The 4\textsuperscript{th} column of the table indicates the degree of freedom of the Chi-square approximation. The test \( H_0: \beta_1 = 0 \) and \( H_0': \beta_0 = 0 \) are rejected.

7. Conclusion

Step-stress accelerated life testing is a useful procedure in reliability studies, which allows collecting failure time data of highly reliable units in a short duration of time. Numerous authors have introduced simple step-stress optimum plans for that purpose. To the best of our knowledge, Khamis and Hinggis [11] were the only ones to attempt to design three and more step-stress accelerated life tests.

In this paper, we considered the situation where the whole set of units may not completely fail in a two step-stress experiment because:

(i) the units may be very highly reliable,

(ii) optimum simple step stress plan having some experimental limitations because simple step stress uses only two stresses that may cause the irrelevant cause of failure,

(iii) the amount of stresses used in the two steps may not be sufficient to make the units fail. Therefore, 3-step step-stress plans are recommended.

The numerical study for obtaining the optimum plan under linear model is tabulated in table 2 for various possible values of scale and shape parameters and the numerical study for the quadratic model is under study. The numerical outputs justify that the optimal time for stress changing is being gradually decreased when increasing the scale (depending on stress levels) with possible increase value of shape parameter. Another important result is that the stress changing time between one to another step is decreasing when simultaneous variation in both the shape and scale parameter is present. The life testing experimenter should take care in jumping from one to another stress level provided that there is not much loss of failure time data of test units while changing the steps.

Finally, we are currently working on numerical investigations for the optimum test plan under a quadratic model for the proposed problem. The development of an optimum test plan under both linear and quadratic models using a Bayesian methodology is also under investigation.

References


Appendix

\[ \begin{align*}
\text{A.1} \\
-n + \left( \frac{n}{A} + 1 \right) \left( \sum_{i=1}^{n_1} \frac{t_{i} \theta_{i}^{1}}{1 + t_{i} \theta_{i}^{1}} + \sum_{i=1}^{n_2} \frac{(t_{i} - t_{j}) \theta_{i}^{1} + t_{j} \theta_{i}^{1}}{(1 + (t_{i} - t_{j}) \theta_{i}^{1} + t_{j} \theta_{i}^{1})} \right) \\
\end{align*} \]
\[\begin{align*}
&+ \sum_{j=1}^{n_3} \left( t_{3j} - \tau_2 \right) \theta_3^{-1} + \left( t_2 - t_1 \right) \theta_2^{-2} + \tau_1 \theta_1^{-1} \\
&- n X_1 - n X_2 - n X_3 + \left( \frac{n}{N} + 1 \right) \sum_{j=1}^{n_1} \frac{X_{1j} \theta_1^{-1}}{1 + t_{1j} \theta_1^{-1}} + \sum_{j=1}^{n_2} \left[ \frac{X_{2j} (t_{2j} - t_1) \theta_2^{-2} + \tau_1 \theta_1^{-1}}{1 + (t_{2j} - t_1) \theta_2^{-2} + \tau_1 \theta_1^{-1}} \right] \\
&+ \sum_{j=1}^{n_3} \frac{X_{3j} (t_{3j} - t_2) \theta_3^{-1} + X_3 (t_2 - t_1) \theta_2^{-2} + \tau_1 \theta_1^{-1}}{1 + (t_{3j} - t_2) \theta_3^{-1} + (t_2 - t_1) \theta_2^{-2} + \tau_1 \theta_1^{-1}} = 0
\end{align*}\]

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