Standardization of the Logistic Distribution based on Entropy

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Abstract: In order to define an acceptable equivalence between a normal and a logistic distribution, a common standardized way is by the identification of their two first statistical moments. We propose an alternative method based on equality of their differential entropies, which demonstrates the validity of the usual standardization method.

Keywords: Logistic Distribution, Normal Distribution, Standardization, Moment Method, Shannon Entropy

1. Introduction

Due to the relative simplicity of its cumulative distribution function, the logistic distribution is frequently used as an equivalent normal distribution. That requires an adequate standardization of the logistic distribution, usually based on the identification of their two first statistical moments. We consider another method based on the equality of their differential entropy containing the whole information of each corresponding distribution. In practice, the two methods have the same efficiency.

2. The Classical Method based on the Standard Deviation

We consider a logistic distribution having a location parameter (α) and a scale parameter (β), using the notation in [1]:

\[ F_X(x) = \frac{1}{1 + \exp\left(-\frac{(x-\alpha)}{\beta}\right)} \]  

Introducing into (1) the mathematical expectation \( E(X) = \xi \) and the standard deviation \( \sqrt{\text{Var}(X)} = \sigma \), we arrive at the standard form:

\[ F_X(x) = \frac{1}{1 + \exp\left(-\frac{\sigma(x-\xi)}{\sqrt{3}\sigma}\right)} \]  

This form can be standardised by considering \( \xi = 0 \) and \( \sigma = 1 \) which yields the representation:

\[ F_X(x) = \frac{1}{1 + \exp\left(-\frac{x}{s_1}\right)} \]  

In this expression we have introduced the reduced scale factor:

\[ s_1 = \sqrt{\frac{3}{\pi^2}} \cong 0.5513 \]

Expression (3) can be directly compared to that of the normal standardised distribution \( \Phi(x) \) to assess their proximity \([1, 2]\). This standardization process is based on the equality of the two first statistical moments. The centering is unavoidable for obtaining a
symmetrical distribution and the choice of the standard deviation determines uniquely the
standardized distribution function.

3. An Alternate Method based on Entropy

Another approach is to use the equality of the standardised normal and logistic
distributions entropies. For a continuous distribution, Shannon’s differential entropy [3] is
the expected value of $E_f = -\int_{-\infty}^{\infty} f(x) \cdot \ln f(x) \cdot dx$ (5)

In the case of the standardised normal distribution [2], we obtain:

$$E_N = \ln \left( \frac{2 \cdot \pi \cdot e}{s} \right) \equiv 1.419$$

(6)

And for a logistic distribution [1] such as defined in (3) having a scale factor $s_2$:

$$E_L = \ln \left( \frac{s_2 \cdot e^2}{s} \right) \equiv 2$$

(7)

The equality between (6) and (7) leads to the scale parameter:

$$s_2 \equiv \sqrt{\frac{(2\pi)}{(e^3)}} \equiv 0.5593$$

(8)

It can be noticed that the values of the scale parameters $s_1 \approx 0.5513$ and $s_2 \approx 0.5593$
are very close (within 1.45 %) which means that the logistic variance provides sufficient
information for characterizing this distribution.

4. Equivalence of the Two Approaches

In practice it seems to be hardly any reason to prefer one of the two methods since the
absolute difference between the corresponding cumulative distribution functions is
(for $i = 1 \& 2$):

$$|F_X(x/s_i) - \Phi_X(x)| \leq 0.023$$

(9)

This maximum deviation occurring for ($x \approx 0.7$) can be substantially reduced as
suggested in [1] (p. 6), by a simple numerical adjustment of the scale parameter $s_1$:

$$s_1' = (16/15) \cdot s_1$$

(10)

The authors expressed this adjustment by a simple ratio of integer instead of a more
precise numerical value (probably because within the framework of this problem it is
unnecessary to obtain an excessive accuracy).

Similarly, taking into account (4) and (8), the value of $s_2$ can be adjusted by a minor
correction of (10) such as:

$$s_2' = (16/15) \cdot (s_1/s_2) \approx (41/39) \cdot s_2$$

(11)

Note that the numerical values in (10) and consequently in (11), were obtained by
matching approximately the logistic and normal cumulative distribution functions in a
range limited to $[0 \leq x \leq 2.5]$ (see Figure 1, page 6 of [1]).
The two adjustments lead essentially to the identical approximation at $\left(x \approx 0.55\right)$:

$$\left|F_X \left(x/s^* \right) - \Phi_X \left(x\right)\right| \leq 0.010 \quad (12)$$

5. Conclusion

The method using entropy provides additional justification for the logistic distribution standardization based on the classical moment method. The two concurrent methods lead to similar approximations of the normal distribution because the mean and the variance of the logistic distribution contain practically the whole information defined on its infinite support.

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References


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