Reliability Analysis of $k$-out-of-$n$ Cold Standby Systems with Erlang Distributions

SUPRASAD V. AMARI*
Parametric Technology Corporation, Greensburg, PA 15601 USA
(Received on September 8, 2011, revised on May 21, 2012)

Abstract: Many fielded systems use cold standby redundancy as an effective system design strategy. However, methods for analyzing the reliability of $k$-out-of-$n$ cold standby systems, particularly with components having age-dependent hazard rates, are limited. In this paper, using the concepts of counting processes, we propose an efficient method to evaluate the reliability of $k$-out-of-$n$ cold standby systems. This proposed method considers Erlang distributions for component lives and the effects of switch failures on system reliability. The main advantage of this counting process-based method is that it reduces a complex problem involving multiple integrals into an equivalent simple problem involving one-dimensional convolution integrals. We consider the Erlang distribution for three reasons: (1) it can be used to model either constant or increasing hazard rates, (2) it can be used to approximate several component failure time distributions, and (3) it has well established closed-form expressions for calculating the convolutions that are used in the counting process-based method. We show that all steps involved in finding the reliability of $k$-out-of-$n$ cold standby system using the proposed method are simple. We demonstrate the proposed method and its computational efficiency using a numerical example.

Keywords: cold standby, $k$-out-of-$n$ systems, Erlang distribution, reliability analysis.

1 Introduction

Cold standby redundancy is used as an effective mechanism for improving system reliability [1]. For example, applications of cold standby redundancy can be found in space explosion and satellite systems [2], electrical power systems [3], and telecommunication systems [4]. Cold standby redundancy involves the use of redundant components that are shielded from the operational stresses associated with system operation. Without exposure to those stresses, the likelihood of failure is very low, and assumed to be zero, until the component is required to operate as a substitute for a failed component [1]. When a failure does occur, it is necessary to detect the failure and to activate the redundant component. For a non-repairable system, the failure detection and switching must be accomplished by additional system hardware that would not otherwise be required. Therefore, for analyzing the reliability of cold standby systems, it is important to consider the effects of switching failures [5, 6].

Although cold standby redundancy has several important applications, the methods for accurately analyzing the reliability of these systems, particularly with components having age-dependent hazard rates, are limited [7, 8]. Closed-form expressions for the reliability of $k$-out-of-$n$ cold standby systems with exponentially distributed component lives are available in the standard reliability textbooks [9, 10].

*Corresponding author’s email: samari@ptc.com
Coit [1] analyzed the reliability 1-out-of-n cold standby systems when component lives follow identical Erlang distributions. Morrison and David [11] derived closed-form expressions for 2-out-of-n cold standby systems when component lives follow identical Erlang distributions with shape parameter equals to 2. When component failure times follow a non-exponential distribution and the system requires multiple operating components for its success (k>1), then the successive failures of the k-out-of-n cold standby system do not follow any standard stochastic process [5]. This is because, at any given time during the mission, the system can have multiple working components with different operational ages. Therefore, to calculate the probability of another failure during the mission, the operational ages of all working components must be considered. Hence, the direct evaluation of system reliability considering the sequences of component failures involves multiple integral equations. However, efficiently evaluating the multiple integral equations is still a challenging task [12]. It not only involves huge computational times but also is prone to numerical round-off errors. The inherent complexity of this direct method is described in section 4 using an example of a 2-out-of-4 cold standby system.

To avoid the use of multiple integrals and associated numerical round-off errors in evaluating system reliability, we apply a counting process-based method. This method was first proposed in [11] and later generalized in [5] to handle non-identical components, warm standby systems, and switch failures. According to this method, the k operating components are considered to be at k logical locations. The key concept used in this method is that as long as the system is operating, the failure processes in all logical locations are independent. Therefore, we can analyze each logical location independently, and then combine their state probabilities to find the system state probabilities. Once we find the system state probabilities, we can find the system reliability as the sum of the probabilities of all success states.

In the counting process-based method, we need to find the probability of a given number of failures in a logical location considering that it acts as a 1-out-of-\((n-k+1)\) cold standby system. This calculation involves the computation of convolution integrals. Although this computation is simpler than multiple integrals, it still requires the use of numerical integration methods for general component failure time distributions. To avoid the explicit use of numerical integrations, we consider Erlang distributions for component failure times. This is because the convolution of two Erlang distributions (with the same scale parameters) is itself an Erlang distribution [9]. Further, Erlang distribution has a closed-form expression for the cumulative distribution function [10]. The main advantage of considering the Erlang distribution is that it can be used as a good approximation or replacement for several component failure distributions [1]. In addition, it can be used to model either constant or increasing hazard rates [9, 10]. In this paper, considering Erlang distributions for component lives, we propose and demonstrate the counting process-based method for analyzing k-out-of-n cold standby systems. This method also considers the effects of switch failures on system reliability.

### 2 Erlang Distribution

The Erlang distribution has a probability density function (pdf) of form [10, 13]:

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\lambda x) \quad \text{for } x > 0$$  \hspace{1cm} (1)

The parameter \(\alpha\) (a positive integer) is called the shape parameter and the parameter \(\lambda\) is called the rate parameter. An alternative, but equivalent, parameterization uses the scale parameter \(\theta\) which is the reciprocal of the rate parameter (i.e., \(\theta = 1/\lambda\)).
Reliability Analysis of $k$-out-of-$n$ Standby Systems with Erlang Distributions

$$f(x) = \frac{x^{\alpha-1} \exp\left(-\frac{x}{\theta}\right)}{\theta^\alpha \Gamma(\alpha)} \quad \text{for } x > 0$$

(2)

Here, $\alpha > 0$ is an integer and $\theta = 1/\lambda > 0$. The distribution, like the Weibull distribution, includes the exponential as a special case ($\alpha = 1$). The hazard rate of the Erlang distribution is constant (time-independent) for $\alpha = 1$ and monotonically increasing for $\alpha > 1$ [10]. Erlang distribution has several important applications in reliability engineering. It can be expressed as the distribution of the sum of independent and identically distributed exponential random variables. Therefore, it is used as the time to failure distribution of 1-out-of-$n$ ($n = \alpha$) cold standby system with perfect switches and exponentially distributed component lives with hazard rate $\lambda$. The cumulative distribution function (cdf) of the Erlang distribution has the following closed-form expression:

$$F(x; \alpha, \lambda) = 1 - \exp(-\lambda x) \left( \sum_{i=0}^{\alpha-1} \frac{(\lambda x)^i}{i!} \right)$$

(3)

To evaluate this function, we need not compute the new factorial and the power function for each $i$. Instead, we can use the following recursive relationships in equation (4):

$$T_0 = 1 \quad \text{and} \quad T_i = T_{i-1} \times (\lambda x)/i \quad \text{for } i = 1 \text{ to } \alpha - 1.$$  

$$F(x; \alpha, \lambda) = 1 - \exp(-\lambda x) \left( \sum_{i=0}^{\alpha-1} T_i \right)$$

(4)

In addition, the cumulative distribution function of the Erlang distribution can be found using the Poisson distribution [12, 13]. Further, the convolution of two Erlang distributions with the same rate parameter, but possibly with different scale parameters, itself is an Erlang distribution. Specifically, the convolution of $F(x; \alpha_1, \lambda)$ and $F(x; \alpha_2, \lambda)$ is $F(x; \alpha_1 + \alpha_2, \lambda)$. In this paper, we utilize this special property to simply the system reliability evaluation.

3 System Description and Assumptions

The proposed method is based on the following system description and assumptions:

- There are a total of $n$ identical components in the system.
- Initially $k$ components are operating, and the remaining $(n-k)$ components are in cold standby.
- Components cannot fail while they are in the standby mode. In other words, the hazard rate of a component in the standby mode is zero.
- The lifetime (failure time) of a component in operation follows an Erlang distribution.
- Immediately upon the failure of an operating component, the component is replaced by one of the standby components in the queue.
- Switches are used to replace the failed component with one of the standby components in the queue, and the switches themselves can fail to operate on demand.
- The replacement of the component is successful only if the switching mechanism is successful.
- The system is operational during the mission when there is $k$ operating components.

4 A Direct Method

The reliability of $k$-out-of-$n$ cold standby system can be evaluated using a direct method based on sequence of failure events. However, this method is computationally
inefficient. To demonstrate the complexity of the direct method, we consider a 2-out-of-4 cold standby system with perfect switches. The system has a total of 4 components, and it will be in operation as long as there are two good components. In other words, the system reaches a failed state at the event of third component failure. Initially, components 1 and 2 are in operation, and components 3 and 4 are in cold standby. Upon the first component failure due to the failure of either of the working components (component 1 or 2), component 3 will be kept in operation. Upon the failure of the next component, component 4 will be kept in operation. Therefore, the system reaches a failed state due to one of the following sequences of failures:

(1) \( x_1 < x_2 < x_3 \)
(2) \( x_1 < x_2 < x_4 \)
(3) \( x_1 < x_3 < x_2 \)
(4) \( x_1 < x_3 < x_4 \)
(5) \( x_2 < x_3 < x_1 \)
(6) \( x_2 < x_3 < x_4 \)
(7) \( x_2 < x_3 < x_3 \)
(8) \( x_2 < x_3 < x_4 \)

where \( x_i \) is the failure time of component \( i \). It is equivalent to the sum of both operational and standby times of component \( i \) at the time of its failure. All these sequences are disjoint. Hence, if we calculate the probability of each of these sequences occurring within the mission time, we can find the system unreliability as the sum of these probabilities. However, the method has several disadvantages. The first disadvantage of this method is that when \( k > 2 \), the number of sequences with distinct probabilities increases exponentially with \( (n-k+1) \) value even when the components are identical. For the 2-out-of-4 system, when the components are identical, the probabilities of sequences (1), (2), (3), and (4) are equivalent to the probabilities of sequences (5), (6), (7), and (8) respectively. However, we still need to find the probabilities for four distinct sequences: (1), (2), (3), and (4). In general, the number of such distinct sequences increases exponentially. The second disadvantage of this method is that the probability calculation of each of these sequences involves multiple integrals that are difficult to solve. The third disadvantage of this method is that, for each sequence, the failure times of components must be tracked to find valid ranges for the integration limits. These are explained further by developing the equations for each of the failure sequences.

Let \( t_i \) be the operational time of component \( i \) at the time of its failure. For sequence (1), we have: \( t_1 = x_1, t_2 = x_2, \) and \( t_3 = x_3 \). The last event in this sequence occurs at \( x_3 \). Hence, the sequence can occur within the mission time \( t \) when \( x_3 < t \). The graphical representation of this sequence is shown in Figure 1.

![Figure 1: Graphical Representation of Sequence 1](image_url)

Let \( Q_i(t) \) be the probability of sequence \( i \) occurring within the mission time. To calculate this probability, for each sequence, we should determine the valid ranges for the operational times of the components. For sequence (1), we have: \( x_1 < x_2 < x_3 \). Therefore,
valid ranges for the operational times associated with this sequence are:

- \( 0 < t_1 < t \)
- \( t_1 < t_2 < t \)
- \( t_2 - t_1 < t - t_3 \)
- \( t_4 > t_3 = t_1 + t_3 - t_2 \)

Hence, the probability of this sequence occurring within the mission time is:

\[
Q_3(t) = \int_0^t f(t_1) \cdot f(t_2) \cdot f(t_3) \cdot f(t_4) dt_1 dt_2 dt_3 dt_4 \tag{5}
\]

where \( f(t) \) is the pdf of failure time of component \( i \). This equation can be simplified as:

\[
Q_3(t) = \int_0^t f(t_1) \cdot f(t_2) \cdot f(t_3) R(t_4 + t_1 - t_2) dt_1 dt_2 dt_3 \tag{6}
\]

where \( R(t) \) is the reliability function of component \( i \). When all components are statistically identical with common pdf, \( f(t) \), and reliability function, \( R(t) \), we have:

\[
Q_3(t) = \int_0^t f(t_1) \cdot f(t_2) \cdot f(t_3) R(t_1 + t_2 - t_3) dt_1 dt_2 dt_3 \tag{7}
\]

Similarly, the graphical representations of sequences (2) through (4) are shown in Figures 2 through 4. The corresponding formulas for \( Q_3(t) \) are shown in equations (8) through (10).

![Figure 2: Graphical Representation of Sequence 2](image)

From Figure 2, the probability of sequence (2) is:

\[
Q_2(t) = \int_0^t f(t_1) \cdot f(t_2) \cdot f(t_3) \cdot f(t_4) R(t_4 - t_2) dt_1 dt_2 dt_3 \tag{8}
\]

![Figure 3: Graphical Representation of Sequence 3](image)

From Figure 3, the probability of sequence (3) is:

\[
Q_3(t) = \int_0^t f(t_1) \cdot f(t_2) \cdot f(t_3) \cdot f(t_4) R(t_2 - t_1) dt_1 dt_3 dt_4 \tag{9}
\]
The effects of switch failures are considered in subsection 5.3.

In this section, we describe the basic concepts and theoretical background of the proposed method for evaluating the reliability of k-out-of-n cold standby systems. The effects of switch failures are considered in subsection 5.3.

Figure 4: Graphical Representation of Sequence 4

From Figure 4, the probability of sequence (4) is:

\[ Q_4(t) = \int_0^t f(t_1) \left( \int_0^{t_1} f(t_2) \left( \int_0^{t_2} f(t_3) \right) R(t_1 + t_2 + t_3) dt_1 dt_2 dt_3 \right) \]  

(10)

Further, when the components are identical, we have: \( Q_d(t) = Q_d(t) \), \( Q_d(t) = Q_d(t) \), \( Q_d(t) = Q_d(t) \), and \( Q_d(t) = Q_d(t) \). Once we compute these probabilities, we can find the system reliability as shown in equation (11).

\[ R_{sys}(t) = 1 - \sum_{i=1}^n Q_i(t) \]  

(11)

The evaluation of \( R_{sys}(t) \) requires computing \( Q_d(t) \), which involves multiple integrals as shown in equations (7) through (10). These computations not only require huge computational times but also are prone to numerical round-off errors. Therefore, the direct method is not practical for evaluating the reliability of k-out-of-n cold standby systems.

5 Proposed Method

In this section, we describe the basic concepts and theoretical background of the proposed method for evaluating the reliability of k-out-of-n cold standby systems. Further, we provide the simplified formulas when the component lives follow Erlang distributions. The effects of switch failures are considered in subsection 5.3.

5.1 Basic Concepts

The proposed method is based on a counting process. In this method, we assume that the operating components are kept at k logical locations or positions. At any location, after the failure of the operating component, it is replaced by a standby component. Therefore, the total number of failures in the system is the sum of the failures at all locations. The probability of a given number of failures at each location is calculated assuming that the failure process at each location is independent of the failure processes (number of failures) at other locations. Because of this independence assumption, the computation of these probabilities becomes simple. Using these probabilities, we calculate the probability of a given number of failures in the whole system. When the switches are perfect, the system is operational as long as there are k good components. In other words, the system is considered to be operating if the total number of failures in the system is less than or equal to \( n-k \). Because we already calculated the probability of a given number of failures (say \( i \) failures) in the system, we can calculate system reliability by adding these probabilities for all failures of component failures, i.e., \( i = 0 \) to \( n-k \).

The key assumption that simplifies the system reliability evaluation is the independence of the failure processes at different logical locations. Therefore, it is important to understand the validity of this assumption to appreciate and accept the proposed method. Strictly speaking, the failure processes at different logical locations are not independent of each other because they all share the common pool of standby
components (spares). However, such a dependency needs to be considered only if there is a shortage of spares. As long as there is no shortage of spares, the failure processes at different logical locations are independent. In the counting process-based method, we consider only those cases where there is no shortage of spares. Hence, the independence assumption used in the proposed method is valid.

5.2 System Reliability Analysis

In this paper, we considered that all components are statistically identical. Thus, all logical locations are not only independent but also identical. Therefore, effectively we need to analyze only one location. Let \( Y_i \) be the failure time of the \( i \)th component used in a logical location. Because the components are identical, each \( Y_i \) has the same failure time distribution, \( F(t; \alpha, \lambda) \). Let \( Z_i \) be the cumulative operational times of all components up to the \( i \)th failure. Therefore,

\[
Z_i = Y_1 + \cdots + Y_i
\]  

(12)

Let \( G_i(t) \equiv \Pr\{Z_i < t\} \) be the cdf of \( Z_i \). It is also equivalent to the probability that there are at least \( i \) failures in the logical location during the mission time. From the properties of the Erlang distribution, \( Z_i \) itself follows the Erlang distribution with a shape parameter \( i\alpha \) and rate parameter \( \lambda \). Hence, we have:

\[
G_i(t) = F(t; i\alpha, \lambda)
\]  

(13)

Note that by definition, we have: \( G_0(t) = 1 \). Let \( p_i \equiv p_i(t) \) be the probability that there are exactly \( i \) failures within the mission time \( t \) in a logical location. Therefore,

\[
p_i = G_i(t) - G_{i-1}(t) = F(t; i\alpha, \lambda) - F(t; (i+1)\alpha, \lambda)
\]  

(14)

Let \( H(m,i) \) be the probability that there are exactly \( i \) failures in the first \( m \) locations. By definition, we have: \( H(1,i) = p_i \). For \( m=2 \) to \( k \), we can calculate \( H(m,i) \) using the following recursive discrete convolution formula:

\[
H(m,i) = \sum_{j=0}^{i} p_j \cdot H(m-1,i-j), \quad i \leq n-k
\]  

(15)

According to equation (15), we can experience \( i \) failures in the first \( m \) locations when there are \( j \) (\( 0 \leq j \leq i \)) failures in the \( m^{th} \) location and \((i-j)\) failures in the previous \((m-1)\) locations. Let \( P_i \equiv P_i(t) \) be the probability that there are exactly \( i \) component failures in the system during the mission time. Note that the total number of component failures in the system is equal to the sum of the failures at all logical locations. Therefore, we have:

\[
P_i = H(k,i)
\]  

(16)

Finally, system reliability is calculated by summing the probabilities of all success states. Because the system is successful when the number of failures is less than or equal to \((n-k)\), we have:

\[
R_{sys}(t) = \sum_{i=0}^{n-k} P_i
\]  

(17)

5.3 Switch Failures

Consider that, at any time the switch is required, there is a constant probability, \( p_{sw} \), that the switch will be successful. In other words, the switch failure probability on request is \((1-p_{sw})\). If there are exactly \( i \) failures in the system, the switch needs to perform its operation successfully for all \( i \) requests. Hence, the switch probability of success for \( i \) requests is \((p_{sw})^i\). In the proposed method, when switches are perfect, we calculate system reliability as the sum \( P_i \) values, where \( P_i \) is the probability of exactly \( i \) failures in the system. When switches are imperfect, we need to multiply these probabilities with switch success probabilities. Hence, system reliability is:
6 Numerical Example

We demonstrate the proposed method using a 4-out-of-8 cold standby system. The failure distribution of the components is Erlang with $\lambda = 0.002$ and $\alpha = 3$. Mission time is $t = 1000$ units of time. Switch success probability on demand is 0.95. In this example, $k = 4$ and $n = 8$. The steps involved in evaluating system reliability are:

1. Calculate $G_i$ values for $i = 0$ to $(n-k+1)$.
   - $G_0 = 1$ and $G_i = F(t;i\alpha\lambda)$ for $i = 1$ to $(n-k+1)$.
   - $F(t;i\alpha\lambda)$ is calculated using equation (3).
2. Using $G_i$ values, calculate $p_i = G_i - G_{i+1}$ for $i = 0$ to $(n-k)$.
3. Calculate $H(m,i)$ for $m = 1$ to $k$ and $i = 0$ to $(n-k)$.
   - Set $H(1,i) = p_i$ for $i = 0$ to $(n-k)$.
   - Calculate $H(m,i)$ for $m = 2$ to $k$ using equation (15).
4. Set $P_i = H(k,i)$ for $i = 0$ to $(n-k)$.
5. Using $P_i$ values, calculate system reliability as in equations (17) and (18).

All calculations involved in the above procedure are simple. The results obtained at each step of the reliability evaluation are provided in Table 1.

Table 1: Reliability Evaluation Steps

<table>
<thead>
<tr>
<th>Step</th>
<th>1</th>
<th>2</th>
<th>3.1</th>
<th>3.2</th>
<th>3.3</th>
<th>3.4</th>
<th>4</th>
<th>5.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>G(t)</td>
<td>p_i</td>
<td>H(1,i)</td>
<td>H(2,i)</td>
<td>H(3,i)</td>
<td>H(4,i)</td>
<td>P_i</td>
<td>$P_i(p_{sw_i})$</td>
</tr>
<tr>
<td>0</td>
<td>1.0</td>
<td>0.677</td>
<td>0.677</td>
<td>0.458</td>
<td>0.310</td>
<td>0.210</td>
<td>0.210</td>
<td>0.210</td>
</tr>
<tr>
<td>1</td>
<td>0.323</td>
<td>0.307</td>
<td>0.307</td>
<td>0.415</td>
<td>0.421</td>
<td>0.380</td>
<td>0.380</td>
<td>0.361</td>
</tr>
<tr>
<td>2</td>
<td>0.016</td>
<td>0.016</td>
<td>0.016</td>
<td>0.116</td>
<td>0.213</td>
<td>0.279</td>
<td>0.279</td>
<td>0.252</td>
</tr>
<tr>
<td>3</td>
<td>2.4e-4</td>
<td>2.4e-4</td>
<td>2.4e-4</td>
<td>0.010</td>
<td>0.050</td>
<td>0.106</td>
<td>0.106</td>
<td>0.091</td>
</tr>
<tr>
<td>4</td>
<td>1.4e-6</td>
<td>1.4e-6</td>
<td>1.4e-6</td>
<td>4.1e-4</td>
<td>0.005</td>
<td>0.022</td>
<td>0.022</td>
<td>0.018</td>
</tr>
<tr>
<td>5</td>
<td>3.9e-9</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
</tbody>
</table>

Step 5.2: Reliability (Sum Probabilities) 0.997 0.932

The last row of the table includes the final system reliability values. With perfect switches, system reliability is 0.997. With switch failures on demand, system reliability is reduced to 0.932. The CPU for solving the problem is 4.53E-5 seconds. Refer to [14] for a method to calculate these small CPU times accurately.

7 Conclusions

In this paper, using the concepts of counting processes, we proposed a simple, accurate, and computationally efficient method to find the reliability of cold standby systems when component lives follow Erlang distributions. The proposed method also considers the effects of switch failures on system reliability. The consideration of Erlang distributions allows us to apply this method for analyzing cold standby systems with components having aging dependent hazard rates. The step-by-step procedure of the method is demonstrated using a numerical example. All steps involved in the proposed method are simple and do not include any complex numerical integrations. The CPU time for the reliability evaluation indicates that the proposed method is extremely fast. Therefore, it can be integrated with redundancy optimization algorithms [1, 6] to evaluate system reliability several times in a short time to find optimal system configurations and redundancy levels.
References


Suprasad V. Amari is a Technical Fellow at Parametric Technology Corporation (PTC). He received both his M.S. and Ph.D. in Reliability Engineering from the Indian Institute of Technology, Kharagpur, India. His research interests include hardware and software reliability, fault-tolerant computing, dynamic dependability models, risk assessment, condition-based maintenance, and optimization. He has authored or co-authored 6 book chapters in Springer Handbooks and over 70 research papers published in reputable international journals and conference proceedings. He is an Assistant Editor-in-Chief for the International Journal of Performability Engineering (IJPE), an editorial board member of the International Journal of Reliability, Quality and Safety Engineering (IJRQSE), and a Management Committee member of the Reliability and Maintainability Symposium (RAMS). Additionally, he is a member of the US Technical Advisory Group (TAG) to the IEC Technical Committee on Dependability Standards (TC 56), a member of the ESRA Technical Committee on System Reliability, advisory board member of several international conferences, and a reviewer for several journals and books on reliability and safety. He received the 2009 Stan Oftshun Award from the Society of Reliability Engineers (SRE) and the 2009 William A. J. Golomski Award from the Institute of Industrial Engineers (IIE). He is a senior member of ASQ, IEEE, and IIE. He is a member of ACM and SRE and an ASQ-certified Reliability Engineer.