K-Round Duel with Uneven Resource Distribution

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Abstract: The paper considers optimal resource distribution between offense and defense and among different rounds in a $K$-round duel. In each round of the duel, two actors exchange attacks. Each actor allocates resources into attacking the counterpart and into defending itself against the counterpart's attack. The offense resources are expendable (e.g., missiles), whereas the defense resources are not expendable (e.g., bunkers). Offense distribution across rounds can increase or decrease as determined by a geometric series. The outcomes of each round are determined by a contest success functions which depend on the offensive vs. defensive resources ratio. The game ends when at least one target is destroyed or after $K$ rounds. It is shown that when each actor maximizes its own survivability, then both actors allocate all their resources defensively. Conversely, when each actor minimizes the survivability of the other actor, then both actors allocate all their resources offensively. We then consider two cases of battle for a single target in which one of the actors minimizes the survivability of its counterpart whereas the counterpart maximizes its own survivability. It is shown that in these two cases the minmax survivabilities of the two actors are the same, and the sum of their resource fractions allocated to offense is equal to 1. However, their resource distributions are different. When both actors can choose their offense resource distribution freely, they distribute all offense to the first round. When one actor is constrained to distribute offense resources across multiple rounds, it is not necessarily optimal for the other actor to allocate all offense to the first round. We illustrate how the resources, contest intensities and number of rounds in the duels impact the survivabilities and resource distributions.

Keywords: Survivability, duel, defense, attack, protection, contest intensity, game theory.

1. Introduction

Through history it has been important to distinguish between the offense and defense. For example, Clausewitz [1] argued for classical warfare for the “superiority of defense over attack”: “The defender enjoys optimum lines of communication and retreat, and can choose the place for battle.” The attacker is advantaged by surprise, but gets exposed by leaving fortresses and depots behind through extended operations. The defense gets improved by trench warfare and the machine gun (World War I), and by castles and fortresses with cannon fire from secure locations. Tanks and aviation (World War II) increased the attacker’s advantage. Hausken [2] suggests that the superiority of the defense over the offense may be even larger for production facilities and produced goods than for Clausewitz’s mobile army. Attackers are more advantaged in the cyber era where defenders have to defend everywhere, while attackers can attack at many locations and at many points in time, [3]. Protective shields are examples of defensive measures, while swords exemplify offensive measures.
The September 11, 2001 attack illustrated that major threats today involve strategic attackers. Therefore, determining risk reduction strategies applying reliability theory and assuming a static external threat does not reflect the real situation. This stimulated emergence of research based on the assumption that the defender and the attacker of a system are fully strategic optimizing agents. In earlier research [4] considers the optimal resource allocation for security in reliability systems. They determine closed-form results for moderately general systems, assuming that the cost of an attack against any given component increases linearly in the amount of defensive investment in that component. References [5-6] assume that the defender minimizes the success probability and expected damage of an attack. Bier et al. [6] analyze the protection of series and parallel systems with components of different values. They specify optimal defenses against intentional threats to system reliability, focusing on the tradeoff between investment cost and security. The optimal defense allocation depends on the structure of the system, the cost-effectiveness of infrastructure protection investments, and the adversary's goals and constraints. Levitin [7] considers the optimal element separation and protection in complex multi-state series-parallel system and suggests an algorithm for determining the expected damage caused by a strategic attacker. Patterson and Apostolakis [8] introduced importance measures for ranking the system elements in complex systems exposed to terrorist actions. Michaud and Apostolakis [9] analyzed such measures of damage caused by the terror as impact on people, impact on environment, impact on public image etc.

Bier et al. [10] assume that a defender allocates defense to a collection of locations while an attacker chooses a location to attack. They show that the defender allocates resources in a centralized, rather than decentralized, manner, that the optimal allocation of resources can be non-monotonic in the value of the attacker's outside option. Furthermore, the defender prefers its defense to be public rather than secret. Also, the defender sometimes leaves a location undefended and sometimes prefers a higher vulnerability at a particular location even if a lower risk could be achieved at zero cost. Dighe et al. [11] consider secrecy in defensive allocations as a strategy for achieving more cost-effective attacker deterrence. Zhuang and Bier [12] consider defender resource allocation for countering terrorism and natural disasters.

Hausken and Levitin [13] present a minmax optimization algorithm. The defender minimizes the maximum damage the attacker can inflict thereafter. The defender has multiple defense strategies which involve separation and protection of system elements. The attacker also has multiple attack strategies against different groups of system elements. A universal generating function technique is applied for evaluating the losses caused by system performance reduction. Levitin and Hausken [14] introduce three defensive measures, i.e., providing redundancy, protecting genuine elements and deploying false elements and analyze the optimal resource distribution among these measures in parallel and k-out of-N systems. Hausken and Levitin [15] analyze the optimal resource distribution among the defensive measures in series systems. Levitin and Hausken [16] analyze the optimal offense and defense resources distribution in the case of two consecutive attacks.

This paper considers a situation when two actors fight offensively and defensively (exchange attacks) with each other over $K$ rounds or until one target is destroyed. Each actor determines the optimal balance between the offense and defense which depends on their resources, the contest intensities, and the number of rounds of attack available. Offense distribution across rounds is or can be affected by the logistics of resource delivery, energy levels, weather, etc., which also can affect the number $K$ of rounds. Each
actor’s optimal strategy depends on whether it maximizes its own survivability, minimizes the survivability of the other actor, or whether the actor maximizes its own survivability while its counterpart minimizes this survivability.

Section 2 presents the model. Section 3 analyzes the model. Section 4 considers the optimal choices of offense resource distribution across subsequent rounds. Section 5 lets the number of rounds vary. Section 6 concludes.

Nomenclature

\( r, R \) actors’ resources
\( \rho \) ratio \( r/R \) between actor 1’s and actor 2’s resources
\( x, X \) offense-defense resource distribution parameters
\( a_i, A_i \) resource allocated to \( i \)-th attack
\( K \) number of consecutive attacks
\( s, S \) target survivability (probability of survival in all \( K \) attacks)
\( v_i, V_i \) success probability of the \( i \)-th attack (for even resource distribution \( v_i \equiv v \))
\( p_i, P_i \) probability that the target is destroyed in the \( i \)-th attack
\( q, Q \) attack effort variation factors
\( \mu, m \) contest intensities in attacks against the actors

2. The Model

Two actors participate in a duel in which they repeatedly attack each other. The total number of consecutive attacks is \( K \), unless one actor is destroyed in attack \( i \) and the game ends. The actors have limited resources \( r \) and \( R \). Each actor distributes its resource among the defense (protection) and offense (attack). The distribution is determined by the parameters \( x \) and \( X \) respectively: resources \( xr \) and \( XR \) are allocated to offense and resources \( (1-x)r \) and \( (1-X)R \) are allocated to defense. \( x \) and \( X \) are two free choice variables.

We denote the actor choosing \( x \) as actor 1 and the actor choosing \( X \) as actor 2. We assume that for both actors the offense resources are expendable (missiles) whereas the defense resources are not expendable (bunkers), which means that the actors use the same protection during the series of \( K \) attacks. The offense resources are distributed among \( K \) attacks. Each actor can observe the outcome of each attack and cease the attacks if the counterpart is destroyed.

The offense resources allocated to attack \( i \) are \( a_i \) and \( A_i \) respectively, such that

\[
\sum_{i=1}^{K} a_i = xr \cdot \sum_{i=1}^{K} A_i = XR.
\]

To determine the vulnerability of an attacked target we use the common ratio form contest success function [17-18], \( w = T^h/(T^h + t^h) = 1/[1+(t/T)^h] \), where \( w \) is the probability of target destruction, \( T \) is the attacker’s effort, \( t \) is the defender’s effort, \( \partial w/\partial T > 0 \), \( \partial w/\partial t < 0 \), and \( h \geq 0 \) is a parameter for the contest intensity. When \( h=0 \), \( t \) and \( T \) have equal impact on \( w \) regardless of their size which gives 50% vulnerability. When \( 0<h<1 \), there is a disproportional advantage of exerting less effort than one’s opponent. When \( h =1 \), the efforts have proportional impact on the \( w \). When \( h>1 \), exerting more effort than one’s opponent gives a disproportional advantage. Finally, \( h=\infty \) gives a step function where “winner-takes-all”. The detailed discussion about the meaning of the contest intensity parameter can be found in [19].

In our case we have attacks against two actors in each of the \( K \) duels. The contest intensities in these attacks can be different. We denote the contests intensities in attacks.
against actors 1 and 2 as \( m \) and \( \mu \) respectively. The success probability of actor 1 investing an offense \( a_i \) in attack \( i \) against the defense \((1-X)R\) is

\[
v_i = v(a_i, (1-X)R) = \frac{a_i^\mu}{a_i^\mu + [(1-X)R]^\mu} = \frac{1}{1 + [(1-X)R/a_i]^\mu},
\]

where \( \partial v_i / \partial a_i > 0, \partial v_i / \partial X > 0, \partial v_i / \partial R < 0 \), and \( \mu \geq 0 \) expresses the intensity of the contest when actor 1 is offensive.

Analogously, the success probability of actor 2 investing an offense \( A_i \) in attack \( i \) against the defense \((1-x)r\) is

\[
V_i = v(A_i, (1-x)r) = \frac{1}{1 + [(1-x)r/A_i]^\mu},
\]

where \( \partial V_i / \partial A_i > 0, \partial V_i / \partial x > 0, \partial V_i / \partial r < 0 \), and \( m \geq 0 \) is the intensity of the contest when actor 2 is offensive. The contest intensities can depend on target locations and, therefore, are different for the two attacks.

In order to destroy the counterpart target in the \( j \)-th attack, each actor must survive in all \( j-1 \) previous attacks. The target can be destroyed in the \( j \)-th attack only if it has not been destroyed in any of the \( j-1 \) previous attacks. Therefore the probabilities that the counterpart target is destroyed in the \( j \)-th attack are

\[
p_j = v_j \prod_{i=1}^{j-1} (1-V_i)(1-v_i), \quad P_j = V_j \prod_{i=1}^{j-1} (1-V_i)(1-v_i). \]

The probabilities of targets destruction in \( K \) consecutive attacks are \( \sum_{j=1}^{K} p_j \) and \( \sum_{j=1}^{K} P_j \) respectively.

\( w_j = p_j(1-V_j) \) and \( W_j = P_j(1-v_j) \) express the probabilities that the actor and its counterpart survive the first \( j-1 \) attacks, the actor survives the \( j \)-th attack, and that the counterpart is destroyed in the \( j \)-th attack

\[
w_j = v_j \prod_{i=1}^{j-1} (1-V_i)(1-v_i), \quad W_j = V_j \prod_{i=1}^{j-1} (1-V_i)(1-v_i). \]

The survivabilities \( S \) and \( s \) of targets 1 and 2 are

\[
S = 1 - \sum_{j=1}^{K} p_j = 1 - V_1 - \sum_{j=2}^{K} V_j \prod_{i=1}^{j-1} (1-V_i)(1-v_i),
\]

\[
s = 1 - \sum_{j=1}^{K} P_j = 1 - v_1 - \sum_{j=2}^{K} v_j \prod_{i=1}^{j-1} (1-V_i)(1-v_i). \]

The actors change the amount of resources allocated to each of the \( K \) attacks. The actors allocate efforts \( a_i \) and \( A_i \), respectively, to the first attack and change their efforts according to the geometric progression:

\[
a_i = qa_{i-1}, \quad A_i = QA_{i-1}, \quad \text{for } 1< i \leq K,
\]

such that
\[ \sum_{i=1}^{k} a_i = a_i \frac{q^i - 1}{q - 1} = x_r, \quad \sum_{i=1}^{k} A_i = A_i \frac{Q^i - 1}{Q - 1} = X_R. \] (7)

The parameters \( q \) and \( Q \) determine the strategy of effort variation through the \( K \) sequential attacks: \( q>1, \ Q>1 \) correspond to increasing the attack effort; \( q<1, \ Q<1 \) correspond to decreasing the attack effort; \( q=1, \ Q=1 \) corresponds to even resource distribution across the \( K \) attacks; \( q=0, \ Q=0 \) corresponds to a single attack with \( a_i=r, \ A_i=R \).

For the given resources and effort variation parameters we obtain

\[ a_i = \begin{cases} \frac{x_r q^i - 1}{q^i - 1}, & q \neq 1 \\ \frac{x_r}{K}, \quad q = 1 \end{cases} \]

\[ A_i = \begin{cases} \frac{X_R Q^i - 1}{Q^i - 1}, & A_i = Q^{-i} A_i \\ \frac{X_R}{K}, \quad Q = 1 \end{cases} \] (8)

(9)

The probabilities of target destruction in attack \( i \) are

\[ V_i = v(q^{-i} a_i, (1-X) R) = \frac{1}{1 + \left[ (1-x) R (q^{-i} a_i) \right]^2} = \begin{cases} \left( \frac{1}{1 + \left[ (1-x) R (q^{-i} a_i) \right]^2} \right)^{\frac{1}{2}}, & q \neq 1 \\ \frac{1}{1 + \left[ (1-x) R (q^{-i} a_i) \right]^2}, & q = 1 \end{cases} \] \[ V_i = v(Q^{-i} A_i, (1-x) r) = \frac{1}{1 + \left[ (1-x) r (Q^{-i} A_i) \right]^2} = \begin{cases} \left( \frac{1}{1 + \left[ (1-x) r (Q^{-i} A_i) \right]^2} \right)^{\frac{1}{2}}, & Q \neq 1 \\ \frac{1}{1 + \left[ (1-x) r (Q^{-i} A_i) \right]^2}, & Q = 1 \end{cases} \] (10)

In the subsequent analysis we set up the general equations when \( q \neq 1 \) and \( Q \neq 1 \) and handle the special cases \( q=1 \) and/or \( Q=1 \) by taking the limit as \( q \) and/or \( Q \) approaches 1.

Inserting (10) into (5) gives

\[ S = \left( \frac{1}{1 + \left[ (1-x) (q^i - 1) \rho \right]^2 \frac{X (Q - 1)}{X (Q - 1)}} \right)^{\frac{1}{2}} \]

\[ -\sum_{i=1}^{k} \frac{1}{1 + \left[ (1-x) (Q_i^i - 1) \rho \right]^2 \frac{XQ^{-i} (Q - 1)}{XQ^{-i} (Q - 1)}} \prod_{j=i}^{k} \left( \frac{1}{1 + \left[ (1-x) (Q_j^i - 1) \rho \right]^2 \frac{XQ^{-j} (Q - 1)}{XQ^{-j} (Q - 1)}} \right) \]
The variables $x, V_i, S$ correspond to actor 1 and its object, and $X, v_i, s$ correspond to actor 2 and its object.

3. Analyzing the Model

We consider four cases. First, in section 3.1 each actor maximizes its own survivability (self-interest situation). Second, in section 3.2 each actor minimizes the survivability of the other actor (mutual aggression). In section 3.3 actor 1 minimizes the survivability $s$ while actor 2 maximizes its survivability $s$ (battle for $s$). Conversely, in section 3.4 actor 1 maximizes its survivability $S$ while actor 2 minimizes actor 1’s survivability $S$ (battle for $S$). In the battle for $s$ and in the battle for $S$ both actors focus exclusively on the survival of one of them, ignoring the survival of the other actor. Finally, section 3.5 compares sections 3.3 and 3.4.

3.1 Maximizing $s$ and $S$

Assume that each actor maximizes its own survivability. If actor 1 chooses $x=0$ and actor 2 chooses $X=0$, the target destruction probabilities in (10) are $v_i=V_i=0$ and the survivabilities in (11) are $s=S=1$. No actor has an incentive to deviate unilaterally from this maximum survivability which thus constitutes an optimal solution where both actors are pacifistic. Both actors refrain from attack and focus exclusively on defense.

3.2 Minimizing $S$ and $s$

Assume that each actor minimizes the survivability of the other actor. If actor 1 chooses $x=1$ and actor 2 chooses $X=1$, the target destruction probabilities in (10) are $v_i=V_i=1$ and the survivabilities in (11) are $s=S=0$. No actor has an incentive to deviate unilaterally from this minimum survivability which thus constitutes an optimal solution where both actors are maximally offensive. Both actors refrain from defense and focus exclusively on attack.

3.3 Battle for $s$

When one actor chooses $x$ to minimize $s$, and the other actor chooses $X$ to maximize $s$, we get the two FOCs

$$ s = \frac{\left(1 - X \right) (q^x - 1)}{x (q - 1) \rho} \cdot \frac{1}{1 + \left(1 - X \right) (q^x - 1)} \cdot \frac{1}{x (q - 1) \rho} $$

$$ - \sum_{j=1}^{K} \frac{1}{1 + \left(1 - X \right) (q^x - 1)} \cdot \frac{1}{x (q - 1) \rho} $$

The variables $x, V_i, S$ correspond to actor 1 and its object, and $X, v_i, s$ correspond to actor 2 and its object.
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which in Appendix A are solved to yield

**Proposition 1:** the FOCs (12) hold for any \(q, Q\) and \(K\) when 

\[ x^* = 1 - X^* \]  

(13)

**Proof:** See Appendix A.

Inserting (13) into (10) and (11) gives the minmax solution

\[
\frac{\partial \psi}{\partial x} = \sum_{j=1}^{\mu} \left( 1 - x \right)^{q - 1} \left( q^2 - 1 \right) \left( \frac{\left( 1 - x \right)^{q^2 - 1} \left( q^2 - 1 \right)}{(q - 1) \rho} \right)^{\mu - 1} \frac{\prod_{j=1}^{\mu} \left( 1 - \frac{1 - \left( 1 - x \right)^{q^2 - 1} \left( q^2 - 1 \right)}{(q - 1) \rho} \right)}{\left( q - 1 \right)^{\mu - 1} \left( q - 1 \right)^{\mu - 1}} + 1 \]  

\[
\frac{\partial \psi}{\partial x} = \sum_{j=1}^{\mu} \left( 1 - x \right)^{q - 1} \left( q^2 - 1 \right) \left( \frac{\left( 1 - x \right)^{q^2 - 1} \left( q^2 - 1 \right)}{(q - 1) \rho} \right)^{\mu - 1} \frac{\prod_{j=1}^{\mu} \left( 1 - \frac{1 - \left( 1 - x \right)^{q^2 - 1} \left( q^2 - 1 \right)}{(q - 1) \rho} \right)}{\left( q - 1 \right)^{\mu - 1} \left( q - 1 \right)^{\mu - 1}} + 1 \]  

\[
\frac{\partial \psi}{\partial x} = \sum_{j=1}^{\mu} \left( 1 - x \right)^{q - 1} \left( q^2 - 1 \right) \left( \frac{\left( 1 - x \right)^{q^2 - 1} \left( q^2 - 1 \right)}{(q - 1) \rho} \right)^{\mu - 1} \frac{\prod_{j=1}^{\mu} \left( 1 - \frac{1 - \left( 1 - x \right)^{q^2 - 1} \left( q^2 - 1 \right)}{(q - 1) \rho} \right)}{\left( q - 1 \right)^{\mu - 1} \left( q - 1 \right)^{\mu - 1}} + 1 \]  

(12)
3.4 Battle for $S$

When one actor chooses $X$ to minimize $S$, and the other actor chooses $x$ to maximize $S$, we get the two FOCs

$$\frac{\partial S}{\partial x} = \frac{\left(\left[\prod_{i=1}^{n} \left(1 - \frac{1-x}{(2-1)x}\right)^{x} \right]^* \cdot \frac{1}{1-x} \right)^* - \left(\left[\prod_{i=1}^{n} \left(1 - \frac{1-x}{(2-1)x}\right)^{x} \right]^* \cdot \frac{1}{1-x} \right)}{\left(\prod_{i=1}^{n} \left(1 - \frac{1-x}{(2-1)x}\right)^{x} \right)^* \cdot \frac{1}{1-x} \cdot \frac{1}{1-x} \cdot \frac{1}{1-x}}$$

which (Appendix A) also gives $x^* = 1 - X^*$ as in (13), and hence $v^*_i, V^*_i, s^*$, and $S^*$ are the same as in (14), but $X^*$ and $x^*$ are not the same.

3.5 Comparing Battle for $s$ and Battle for $S$ Solutions

We define $X^*(opt \ s)$ as the optimal $X$ in the battle for $s$, and $X^*(opt \ S)$ as the optimal $X$ in the battle for $S$. Section 3.3 and 3.4 show that when both actors focus exclusively on the survival of one of the actors, maximizing it and minimizing it, respectively, ignoring the survivability of the other actor, then the survivabilities of the two actors are the same, though their allocations to offense and defense are different. This means that when optimizing $s$, actor 2 chooses $X^*$ from section 3.3 and actor 1’s best response is $x^* = 1 - X^*$. When optimizing $S$, if actor 2 chooses $X^*$ from section 3.3, then actor 1 responds with some $x^* = 1 - X^*$, which gives $S$ larger than in (14). Actor 2 can achieve lower $S$ by choosing $X^*$ from section 3.4, and actor 1’s best response is $x^* = 1 - X^*$, which gives (14) again. That two different games and two different strategies cause the same survivabilities has to do with $x^* = 1 - X^*$ and the symmetry of the two situations. Both actors have different focuses in the two situations, though in a manner causing the same survivabilities. Sections 3.3 and 3.4 are linked as follows: if we replace $m$ with $\mu$, $\mu$ with $m$ and $\rho$ with $1/\rho$, the situation becomes symmetrically opposite to the actors and, therefore, $X^*(opt \ S)$ in section...
3.4 is equal to \( x^*(\text{opt } s) = 1 - X^*(\text{opt } s) \) in section 3.3. After such replacement in section 3.4, we get \( X^*(\text{opt } s) + X^*(\text{opt } S) = 1 \).

For \( m = \mu \) and \( \rho = 1 \) (which corresponds to total symmetry in the duel), solving the FOCs in sections 3.3 and 3.4 for \( K = 2 \) and \( q = Q = 1 \) gives

\[
X^*(\text{opt } s) = \frac{1}{2 + 2^{-\mu} + 2^{1+\mu}}, X^*(\text{opt } S) = 1 - \frac{1}{2 + 2^{-\mu} + 2^{1+\mu}}, X^*(\text{opt } S) + X^*(\text{opt } s) = 1,
\]

\[
X^*(\text{opt } S) = X^*(\text{opt } s) = \frac{2^{-\mu} + 2^{1+\mu}}{2 + 2^{-\mu} + 2^{1+\mu}} > 0.
\]

Hence in the symmetric situation the actor that fights for survival must always be less aggressive, by spending more resources on defense, than the actor that fights for destruction of its counterpart.\(^1\)

4. Optimal \( q \) and \( Q \)

This section considers a two period minmax game where actor 1 chooses \( q \) first to minimize the maximum possible \( s^* \) in (14) that actor 2 can cause thereafter by choosing \( Q \) (assuming that the actors also optimize their \( x \) and \( X \)). That is, the optimal actor 1 strategy \( q \) is the solution of a minmax game in which actor 1 chooses \( q \) that minimizes \( s^* \) given that for any \( q \) the attacker chooses \( Q \) that maximizes \( s^* \). The optimal values of \( q \) and \( Q \) can be obtained by the following enumerative procedure.

1. Assign \( s^*_\text{min} = 1 \);
2. for each \( q = q_{\text{min}}, q_{\text{min}} + \Delta, \ldots, q_{\text{max}} \)
   2.1.1. assign \( s^*_\text{max} = 0 \);
   2.1.2. for each \( Q = Q_{\text{min}}, Q_{\text{min}} + \Delta, \ldots, Q_{\text{max}} \) obtain \( s^*(q, Q) \) using (14) and if \( s^*_\text{max} < s^*(q, Q) \) assign \( s^*_\text{max} = s^*(q, Q) \); \( Q_{\text{opt}} = Q \);
   2.1.3. if \( s^*_\text{max} < s^*_\text{min} \) assign \( s^*_\text{min} = s^*_\text{max} \), \( Q^* = Q_{\text{opt}} \), \( q^* = q \).

The solutions obtained using this procedure show that for any \( K, m, \mu \) and \( \rho \), \( q^* = Q^* = 0 \) which means that the duel lasts \( K = 1 \) round. Hence both actors use their resources in their entirety in the first round. If one actor were to save resources for the second round, he will lose against an actor allocating all resources in the first round.

To understand the minmax solution \( q^*, Q^* \) consider Figure 1 which plots \( s^* \) and \( S^* \) as functions of \( Q \) for various \( q \) when \( K = 5 \) and \( m = \mu = \rho = 1 \).

**Figure 1:** \( s^* \) and \( S^* \) as functions of \( Q \) for various \( q \) when \( K = 5 \) and \( m = \mu = \rho = 1 \)

\(^1\) Numerical tests suggest that \( X^*(\text{opt } S) - X^*(\text{opt } s) > 0 \) for general \( K, m, \mu \) and \( \rho \), but we have not been able to prove that.
Observe that with the decrease of $q$, $S^*$ increases and the curve changes the shape such that the local minimum disappears (the point when $S^*$ is minimal shifts to $Q^*=0$ with the decrease of $q$). Analogously for $s^*$: with the decrease of $q$, $s^*$ decreases and the curve changes the shape such that the local maximum disappears (the point when $s^*$ is maximal shifts to $Q^*=0$ with the decrease of $q$). Consequently, $Q^*$ can differ from 0 for fixed $q$. However, when $q$ and $Q$ are free, $Q^*=q^*=0$.

Figure 1 shows that if one actor is constrained to choose $Q$ (or $q$) larger than 0, it is not necessarily optimal for the other actor to choose $q=0$ (or $Q=0$). Consider for example a duel with $K=2$ and $\mu=1$ in which one actor distributes its resource evenly between two rounds ($Q=1$). Inserting $Q=1$ into $s^*$ in (14) (using L’Hopital’s rule) and solving the FOC $\partial s^*/\partial q=0$ gives

$$ q^* = \frac{2^{2}\rho^{m+2}(2+\rho+2^{m}+\rho^{m})-1-\rho}{1+2\rho+\rho^{2}+2^{m}\rho^{m+1}} $$

which can be larger than 0. The second order condition is satisfied at optimum, $\partial^2 s^*/\partial q^2>0$.

5. Constrained Resource Distribution

In some cases $q$ and/or $Q$ are determined exogenously. Commonly the logistics of resource delivery is such that resources cannot be delivered in their entirety before the first round, affected by production rates, ammunition supply rates, etc. And, even if such delivery were possible, compiling a huge resource into one decisive blow against an opponent may be impossible. A common scenario can be much resource allocation in the first round, and more modest resource allocation in subsequent rounds as supplies come in ($q<1$ and/or $Q<1$). Conversely, actors eager to duel before resources are in place can cause increasing resource allocation as the duel progresses ($q>1$ and/or $Q>1$). For human actors, e.g., in a boxing duel, confining all energy into one single blow is impossible. Energy levels commonly decrease as the boxing match progresses, but can also increase due to psychological or physiological reasons. $q$ and $Q$ can also be affected by weather, daylight, etc.

The following are examples of minmax game solutions when the offensive resource distribution parameters for both actors $q$ and $Q$ are determined exogenously. Figure 2 plots $X^*(opt\ s)$, $S^*(opt\ S)$, $s^*$, and $S^*$ as functions of $\rho$ for $K=5$, $q=2$, $Q=0.5$ and different $m$ and $\mu$. Actor 1 suffers high $s^*$ when $\rho$ is low, and $s^*$ decreases as actor 1 becomes more resourceful. $s^*$ decreases more quickly when $m$ is large. The high $q>Q$ causes actor 1 to suffer high $s^*>0.5$ for $\rho=1$ in all cases except $\{m=2, \mu=0.5\}$ which causes high (inverse U shaped) offense $X^*(opt\ s)$ by actor 2 to take advantage of the high $m$. When actor 1 is resourceful, its least preference is the case $\{m=0.5, \mu=2\}$ which causes high $s^*$ since actor 1’s high $q$ prevents it from taking immediate advantage of the high $\mu$, and the low $m$ gives actor 2 with little resources an advantage in the egalitarian contest.
When one of the actors has full information about the offensive resource distribution of its counterpart, this actor can optimize its own offensive resource distribution by minimizing (maximizing) the corresponding survivability (14). For example, 

\[ Q^*_{\text{opt}} = \arg \max_{0 \leq q \leq 1} \{s(m, \mu, \rho, q)\} \quad \text{and} \quad Q^*_{\text{opt}} = \arg \min_{0 \leq q \leq 1} \{S(m, \mu, \rho, q)\}. \]

Figure 3 plots \( Q^*_{\text{opt}} \), \( X^*_{\text{opt}} \), \( s^*_{\text{opt}} \), \( S^*_{\text{opt}} \), \( Q^*_{\text{opt}} \), \( X^*_{\text{opt}} \), \( s^*_{\text{opt}} \), and \( S^*_{\text{opt}} \) as functions of exogenously given \( q \) for \( K=5 \), \( m=\mu=1 \) and different \( \rho \).

Actor 2 chooses optimal \( Q^*_{\text{opt}} \) equal to zero when \( q \) is low, consistently with \( q^* = Q^* = 0 \) when both \( q \) and \( Q \) are chosen freely, and increases \( Q^*_{\text{opt}} \) gradually as \( q \) increases, though \( Q^*_{\text{opt}} \) is substantially lower than \( q \). When actor 2 is inferior (\( \rho = 2 \)), it relies on a large first round attack (\( Q^* = 0 \)) despite \( q \) increasing toward 1. When actor 2 is superior (\( \rho = 0.5 \)), and it seeks to minimize the survivability \( S \) of actor 1, it increases its \( Q^*_{\text{opt}} \) to succeed against actor 1 with large \( q \) also in subsequent rounds of the duel.
6. Variable Number of Rounds

Factors that affect \( q \) and \( Q \), such as the logistics of resource delivery, energy levels, weather, can also affect the number \( K \) of rounds. Additionally, \( K \) can be affected by culture, routine, and developed practice. In sports, e.g., boxing, 12 rounds are common. In many local conflicts two sides have limited time (until intervention of UN or superpowers) to exchange the attacks, which limits the number of rounds of the duels.

Figure 4 plots \( X^*(\text{opt } s), X^*(\text{opt } S), s^*, \) and \( S^* \) as functions of \( K \) for \( m=\mu=\eta=Q=1 \) and different \( \rho \).

![Figure 4: Graphs showing the relationship between K and X*(opt s), X*(opt S), s*, and S* for different values of \( \rho \).](image)

Figure 4 shows that the increase of the number of rounds \( K \) is beneficial to the actor that has the resource superiority. However, with increase of \( K \) the outcome of the duel becomes almost insensitive to the number of rounds in the duel. With \( K=1 \) rounds, actor 2 chooses \( X^*(\text{opt } s)=0 \), thus focusing on defense only to maximize its own survival \( s^* \). With \( K=1 \) rounds, actor 2 chooses \( X^*(\text{opt } S)=1 \), thus focusing on attack only to minimize the survival \( S^* \) of actor 1. As \( K \) increases, \( X^*(\text{opt } s) \) increases concavely, and more pronounced when actor 2 is most resourceful, and \( X^*(\text{opt } S) \) decreases convexly, and less pronounced when actor 2 is most resourceful.

Figure 5 plots \( X^*(\text{opt } s), X^*(\text{opt } S), s^*, \) and \( S^* \) as functions of \( K \) for \( m=\mu=\rho=1 \) and different \( q \) and \( Q \). It can be seen that when \( q=Q \) the actors’ survivabilities are almost insensitive to the number of duels \( K \). When \( q\neq Q \) and one of the actors increases the attack effort whereas its counterpart decreases the attack effort through the \( K \) rounds, the actor with increasing attack effort suffers from increase of the number of rounds \( K \). This finding is consistent with our earlier result that the actors prefer \( q^*=Q^*=0 \). When the number of rounds exceeds 10, its influence on the outcome of the duel becomes negligible. The most prominent increase of \( X^*(\text{opt } s) \) occurs for the case \( \{q=2, Q=0.5\} \). Actor 2 maximizes its own survival \( s^* \), knows that actor 1 increases its attack effort through the \( K \) rounds, and chooses the offensive to prevent actor 1 from becoming a future threat. For the opposite reason, \( X^*(\text{opt } s) \) is low for the opposite case \( \{q=0.5, Q=2\} \). As in section 3.5, we always have \( X^*(\text{opt } S)>X^*(\text{opt } s) \).

6. Conclusion

Two actors distribute their resources between offense and defense in \( K \) repeated duels. The defense resources are not expendable (e.g., bunkers). The offense resources are expendable (e.g., missiles) and are distributed unevenly across \( K \) attacks, determined by a geometric series which allows for increasing or decreasing attacks through the \( K \) rounds. Offense distribution across rounds is or can be affected by the logistics of resource delivery, energy levels, weather, etc., which also can affect the number \( K \) of rounds. The outcomes of the two duels in each round are determined by a contest success function which depends on the offensive and defensive resources. The game ends when at least one target is destroyed or after \( K \) rounds.
We have shown that when each actor maximizes its own survivability, then both actors allocate all their resources defensively. Conversely, when each actor minimizes the survivability of its counterpart, then both actors allocate all their resources offensively. We then consider two cases of battle for a single target in which one of the actors minimizes the survivability of its counterpart whereas the counterpart maximizes its own survivability. In these two cases the two actors’ minmax survivabilities are the same, and the sum of their resource fractions allocated to offense equals 1. However, their allocations to offense and defense in the two cases are different. When the actors are equally resourceful, the two contest intensities are equal, and the offense resources are distributed evenly across two attacks, then the actor that fights for the destruction of its counterpart always allocates more resources to offense.

When both actors freely choose their offense resource distribution, they distribute all offense to the first round. When one actor is constrained to distribute offense resources across multiple rounds, it is not necessarily optimal for the other actor to allocate all offense to the first round. We suggest a numerical procedure for optimizing each actor’s offense resource distribution given the predetermined offense resource distribution of its counterpart.

We have demonstrated the methodology of analyzing the minmax solutions to show how the resources, contest intensities and number of attacks impact the survivabilities and allocations to offense and defense. It can be seen that the increase of the number of attacks in the duel $K$ is favorable for the actor that has resource superiority.

References

Appendix A: (Proof of Proposition 1)

Let $s_k$ be the survivability of target 2 for the case of $K$ duels. Using (5), mathematical induction implies...
\( s_2 = 1 - v_1 - v_2(1 - V_1) \)
\( s_k = 1 - v_1 - \sum_{j=2}^{k} v_j \prod_{i=j}^{k-1} (1 - V_i)(1 - v_i) = 1 - v_1 - \sum_{j=2}^{k-1} v_j \prod_{i=j}^{k-1} (1 - V_i)(1 - v_i) - v_k \prod_{i=k}^{k-1} (1 - V_i)(1 - v_i) \)
\( = s_{k-1} - v_k \prod_{i=k}^{k-1} (1 - V_i)(1 - v_i) \)

\( \frac{\partial s_k}{\partial x} = \frac{\partial s_{k-1}}{\partial x} - \frac{\partial}{\partial x} \left[ v_k \prod_{i=k}^{k-1} (1 - V_i)(1 - v_i) \right] \) for \( k > 2 \).

\( \frac{\partial s_k}{\partial X} = \frac{\partial}{\partial X} \left[ v_k \prod_{i=k}^{k-1} (1 - V_i)(1 - v_i) \right] \) for \( k > 2 \).  

(A1)

Now we have to prove that

1. If \( \frac{\partial s_k}{\partial x} = 0 \) holds when \( x = 1 - X \),  
\( \frac{\partial s_k}{\partial X} = 0 \)  
(A2)

2. If \( \frac{\partial}{\partial x} \left[ v_k \prod_{i=k}^{k-1} (1 - V_i)(1 - v_i) \right] = 0 \) holds when \( x = 1 - X \)  
\( \frac{\partial}{\partial X} \left[ v_k \prod_{i=k}^{k-1} (1 - V_i)(1 - v_i) \right] = 0 \)  
(A3)

Inserting (10) into (A1) and (A2) and differentiating \( s_k \) gives

(A4)

\( \frac{\partial s_k}{\partial x} = \left( \frac{(k - 1) [q^{k - 1}]}{(q - 1) x} \right)^{m} \sum_{r=0}^{m} \left( \frac{(x - 1) [q^{k - 1}]}{(q - 1) x} \right)^{r} \left( \frac{(x - 1) [q^{k - 1}]}{(q - 1) x} \right)^{1} \) \( \left( \frac{(x - 1) [q^{k - 1}]}{(q - 1) x} \right)^{1} \) 

Observe term by term that these two expressions are equivalent except that 1-\( x \) in one denominator in the first expression is replaced by \( X \) in the corresponding denominator in the second expression, and that \( x \) in a second denominator in the first expression is replaced by 1-\( X \) in the corresponding denominator in the second expression. Consequently, if (A2) holds then \( x = 1 - X \).

Inserting (10) into (A3) and differentiating gives
The two denominators are equivalent when $x = 1 - \lambda$, and the second terms in the two numerators are equivalent when $x = 1 - \lambda$. The first terms in the two numerators are equivalent when

\[
\left(\frac{1}{(1 - x)(q - 1)\rho}\right)^\mu 1 \left[\frac{(1 - X)q^{(q - 1)}(q^q - 1)}{(q - 1)\rho}\right] = 0
\]

which involves products of the two differentiations

\[
\frac{\partial}{\partial x} \left(1 - X \right) \frac{q^{(q - 1)}}{x} = \mu \left(1 - X \right) \frac{q^{(q - 1)}}{x}
\]

and

\[
\frac{\partial}{\partial x} \left(1 - X \right) \frac{q^{(q - 1)}}{x} = -m \left(1 - X \right) \frac{Q^{(Q - 1)}\rho}{(Q - 1)}
\]

which are also equivalent when $x = 1 - \lambda$, and hence (A3) holds when $x = 1 - \lambda$. Proving that (A2) and (A3) hold when $x = 1 - \lambda$, we proved that FOCs (12) hold when $x = 1 - \lambda$ for any parameters $K, \rho, m, \mu, Q, q$. In the similar manner one can prove that if FOCs (15) hold, then $x = 1 - \lambda$ for any parameters $K, \rho, m, \mu, Q, q$. 

\[
\theta (y_1, \ldots, y_n) = \frac{\partial}{\partial x} \left(\frac{1}{(q - 1)\rho}\right) 1 \left[\frac{(1 - X)q^{(q - 1)}}{(q - 1)\rho}\right] = \mu \left(1 - X \right) \frac{q^{(q - 1)}}{x}
\]

\[
\theta (y_1, \ldots, y_n) = \frac{\partial}{\partial \lambda} \left(\frac{1}{(q - 1)\rho}\right) 1 \left[\frac{(1 - X)q^{(q - 1)}}{(q - 1)\rho}\right] = -m \left(1 - X \right) \frac{Q^{(Q - 1)}\rho}{(Q - 1)}
\]