Analytical Uncertainty Propagation for Availability Assessment of Stochastic Multi-state Systems

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Abstract: The proposed method generalizes uncertainty propagation methods for complex stochastic multi-states system modeling. Until now, analytical methods for uncertainty propagation are only used in simple system modeling as fault tree model. These methods are usually based on moment calculation or variance evaluation. In complex system cases or when complex mathematic factors affect the uncertainties, asymptotic calculation or Monte Carlo simulation are often used to estimate the resulting availability. The proposed method is based on an analytical calculation of the probability of the system states to propagate analytically the uncertainty in the model. It allows a better assessment of the availability of stochastic multi-state systems. The analytical methods used are based on calculus of conjoint probability density function, Universal Generating Function model and associated operators. The method consists in an analytical propagation of uncertainty in the model and an output uncertainty assessment. Particularly, it is able to deduce analytically the output availability of the system with the associated uncertainty. For complex UGF modeling, the analytical calculation to the sub-systems and Monte Carlo simulation on the resulting UGF system model are applied. Consequently, the computing time for the simulation could be drastically reduced.

Keywords: Universal Generating Function, data uncertainty propagation, uncertainty modeling, probabilistic models, Markov processes, Monte Carlo simulation

1. Introduction

The uncertainties affecting the input data of a stochastic multi-state system (MSS) have to be considered in a risk analysis to assess the system availability. It must be distinguished the uncertainty assessment of input data [9] and the propagation process of the uncertain data in reliability modeling [15]. This article is focused on the propagation process using an appropriate model for the uncertainties and based on an adapted MSS model to assess its availability. The uncertainty assessment is not studied in the present work, and consequently the statistics treatments and possible Bayesian approaches, expert judgments treatment, etc. To take into account this context, different approaches can be used to model the uncertain data. The main two approaches are the probabilistic ones (probability density propagation) and the possibilistic ones (propagation using fuzzy logic and fuzzy numbers [10], Dempster-Shafer theory, etc.). There exist also some hybrid approaches based on uncertain probability theories but not yet applied to MSS models [14]. In terms of the possibilistic approaches, fuzzy theory has been applied to MSS using the UGF and Markovian models, but no study has been led in the probabilistic approach using this kind of model.
In [2], authors propose fuzzy values to model uncertainty about performance levels of components and associated probabilities in a UGF (Universal Generating Function) model. Fuzzy theory operators have been used to obtain performance level of the system. The reliability and performance assessment for fuzzy multi-states elements is studied in [3]. This article focuses on Markovian multi-states component using fuzzy sets to model uncertainty about the component performance levels and associated probabilities. Also, fuzzy values have been studied to present the uncertain component states’ level and their associated probabilities in [8]. The UGF function and the UGF operators take into account triangular fuzzy numbers from the fuzzy theory to assess MSS availability. It can be noticed that these works solve partially the problem of availability assessment of Markovian multi-states system with uncertain data as it is proposed in this article.

A component according to a stochastic process has different states with a certain level of performance. Transition (failure) rates characterize the component degradation from a given level of performance to lower performance levels (in the proposed approach, Markovian processes have been used, see [2] for details about the degradation modeling with these processes). The uncertainty on the failure rates affects the knowledge on the component state and introduces an uncertainty on the probability of the component to be in this state. A stochastic multi-state system (MSS) is composed of stochastic multi-states components. A stochastic multi-states system has multi-level of performances that depends on the level of performance of its components. To assess the state probabilities of such system and its availability, all of its components states should be evaluated at first, and should be combined to form the system model (connection of components in a series, in parallel with redundancies, components in a bridge structure). The universal generating function (UGF) [13] [4], applied on a component, modeled its different states taking into account its different performance levels. To model a stochastic multi-state system, the composition operators should be applied over the UGF of individual components. These operators have been defined in [6], and can be recursively applied on two components to obtain the general model of the system (combination of operators). More details about the stochastic multi-state system model are given in [7]. In a given system modeling, uncertainty on the failure rate of a component is propagated into the system and affects the availability assessment of the system [5].

We consider a complex stochastic repairable multi-states system composed of different sub-systems and modeled by UGF. To assess the output probability of different states and the availability of the sub-system taking into account uncertainty on failure rate of its components, the proposed method consists in: modeling each stochastic multi-state component by the UGF and the uncertain failure rate by a usual probability density function (pdf). Then, the uncertainty on each component failure rate is considered as a variation around the mean value of the failure rate according to different probability law (normal, log-normal, uniform…). By applying the composition operators, the uncertainties on the failure rates of each component are integrated and propagated in the whole system model. The uncertainty effect depends on various criteria: value of the failure rate, amplitude of uncertainty on the failure rate, functional / structural influence of the component in the system… The system availability and each state probability are then evaluated analytically.

In the first section, a repairable stochastic multi-states component modeled by UGF is presented. Then, the general method of analytical calculus of probability states and the propagation of uncertainty is detailed. The second section generalizes the method for stochastic repairable multi-states systems. The third section presents an example of multi-states system modeling and probability calculus of its states. The last section is a case...
study composed of two sub-sections: (1) analytical propagation of uncertainty and analytical assessment of availability for two states stochastic components and (2) analytical propagation of uncertainty and analytical assessment of availability for three states stochastic components.

2. Analytic Uncertainty Propagation in a Stochastic Repairable Multi-states Component

A component behavior described by Markov process (stochastic process), is characterized by several states of performance starting from a nominal state having a maximum performance to achieve a state of total failure. The states are memory-less. We note $C_i$ a stochastic repairable component and $p_{ik}$ the probability to be in the state $i$. $C_i$ can degrade from state “i” to state “i-1” according to a failure rate $\lambda_{ik,1}$ (i from 1 to $n$, $n$ being the number of possible component states). The repairable rate $\mu_{ik,1}$ which upgrades the component from state “i-1” to a higher performance state “i” is considered known (see Fig. 1). Its associated uncertainty is modelled by a usual probability function (uniform U, normal N, log-normal LogN, or exponential laws Exp). This uncertainty is characterized by a fixed standard deviation $\sigma$ and the average $m$. Each law $Y$ is written as: $Y [m, \sigma]$, where $m$ is the law symbol, (U for uniform law, etc.). The Universal Generating Function (UGF) is used to model the states of the component $C_i$, which gives ($g_k$ is the performance of the component in the state $i$): $UGF_i = p_{i1}z^{e_{i1}} + p_{i2}z^{e_{i2}} + \ldots + p_{in}z^{e_{in}} + RZ^n$.

![Figure 1: Markov Process of a Repairable $n$-states Component](image)

The probability of each state $p_{ik}$ can be obtained by solving the system “$dS_k$” resulting of Markov process (see Fig. 1) where $dS_k$ is:

$(n)$: $\frac{dp_{ik}}{dt} = -\lambda_{kn,1}p_{ik} + \mu_{kn,1}p_{kn-1}$

$(n-1)$: $\frac{dp_{ik}}{dt} = \lambda_{kn,1}p_{kn-1} - (\lambda_{kn-2} + \mu_{kn-1})p_{kn-2} + \mu_{kn-2}p_{kn-2}$

$\ldots$ $\ldots$

$(2)$: $\frac{dp_{ik}}{dt} = \lambda_{k2}p_{k2} - (\lambda_{k1} + \mu_{k2})p_{k1} + \mu_{k2}p_{k1}$

$(1)$: $\frac{dp_{ik}}{dt} = \lambda_{k1}p_{k1} - \mu_{k1}p_{k1}$

The solution of this system has the form $S_k$:

$(n)$: $p_{kn}(t) = f_n(t, \mu_{k1}, \mu_{k2}, \ldots, \mu_{kn}, \lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kn})$

$(n-1)$: $p_{kn-1}(t) = f_{n-1}(t, \mu_{k1}, \mu_{k2}, \ldots, \mu_{kn}, \lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kn})$

$\ldots$ $\ldots$

$(2)$: $p_{k2}(t) = f_2(t, \mu_{k1}, \mu_{k2}, \ldots, \mu_{kn}, \lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kn})$

$(1)$: $p_{k1}(t) = f_1(t, \mu_{k1}, \mu_{k2}, \ldots, \mu_{kn}, \lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kn})$

where $f_1, f_2, \ldots, f_n$ are functions of variables $\lambda_{ij}$ and where time “$t$” and repairable rate “$\mu$” are considered as known and constant parameters. We consider that functions $f_i$ are monotone piecewise. The method is applied separately on interval where functions $f_i$ are bijective and invertible. To propagate analytically the uncertainty on failure rate in the
Let consider $S_t$ a system of $n$ equations where $(\lambda_{k1}, \lambda_{k2}, ..., \lambda_{kn-1}, \lambda_{kn})$ are the only unknown variables and are independent of time $t$ as considered above. The solution $S_{tk}$ of this system has the form:

\[
\begin{align*}
(n): & \quad \lambda_{kn}(p_{ki}) = h_0(p_{k1}, p_{k2}, ..., p_{kn-1}, p_{kn}) \\
(n-1): & \quad \lambda_{kn-1}(p_{ki}) = h_n(p_{k1}, p_{k2}, ..., p_{kn-1}, p_{kn}) \\
(2): & \quad \lambda_{k2}(p_{ki}) = h_2(p_{k1}, p_{k2}, ..., p_{k_{n-1}}, p_{kn}) \\
(1): & \quad \lambda_{k1}(p_{ki}) = h_1(p_{k1}, p_{k2}, ..., p_{k_{n-1}}, p_{kn})
\end{align*}
\]

Where $p_{ki}$ are the variables (probability $p_{ki} \in [0, 1]$) and where $t$ and $\mu$ are known and constant parameters. Let $det_{k}$ be the determinant of $J_k$ then: \(l(det_{k})^\mu = 11|det_{k}|=\)function of $(\lambda_{k1}, \lambda_{k2}, ..., \lambda_{kn-1}, \lambda_{kn})$. To obtain \(l(det_{k})(p_{ki})^\mu\) in form of $p_{ki}$ we should replace $\lambda_{ki}$ by $\lambda_{ki}(p_{ki})$ in the expression of the determinant so:

\[
l(det_{k})(p_{ki})^\mu = l(det_{k}(h_1(p_{k1}, p_{k2}, ..., p_{kn-1}, p_{kn}), h_2(p_{k1}, p_{k2}, ..., p_{kn-1}, p_{kn}), ..., h_n(p_{k1}, p_{k2}, ..., p_{kn-1}, p_{kn})))^{\mu}
\]

So the probability density function of the states of the component can be obtained as $(i=1, 2, ..., n-1, n)$ with the expression of \(l(det_{k})(p_{ki})^\mu\) given by (1):

\[
pdf_{ki}(p_{k1}, p_{k2}, ..., p_{kn-1}, p_{kn}) = pdf_{k1}(h_1(p_{k1}, p_{k2}, ..., p_{kn-1}, p_{kn}), h_2(p_{k1}, p_{k2}, ..., p_{kn-1}, p_{kn}), ..., h_n(p_{k1}, p_{k2}, ..., p_{kn-1}, p_{kn})), (det_{k})(p_{ki})^\mu
\]

When the component $C_k$ follows a Markov process, its states are independent and so:

\[
pdf_{k1}(\lambda_{k1}, \lambda_{k2}, ..., \lambda_{kn-1}, \lambda_{kn}) = pdf_{k1}(\lambda_{k1}) \cdot pdf_{k2}(\lambda_{k2}) \cdot ... \cdot pdf_{kn}(\lambda_{kn})
\]

Putting (3) in (2) we obtain:

\[
pdf_{ki}(h_1(p_{k1}, p_{k2}, ..., p_{kn-1}, p_{kn})), pdf_{k1}(h_2(p_{k1}, p_{k2}, ..., p_{kn-1}, p_{kn})), ..., pdf_{kn}(h_n(p_{k1}, p_{k2}, ..., p_{kn-1}, p_{kn})), (det_{k})(p_{ki})^\mu
\]

To obtain $pdf_{ki}(p_{ki})$ we should integrate (4) from 0 to 1 on all variables unless $p_{ki}$:

\[
\int_1^{\pi_n} \int_2^{\pi_n} ... \int_{n-1}^{\pi_n} dp_{k1} \cdot dp_{k2} \cdot ... \cdot dp_{k_{n-1}} \cdot dp_{kn}
\]

To obtain the mean $p_{ki}(t)$ we should find the primitive of (5) on variable $\pi_k$ as follows:

\[
p_{ki}(t) = \int \pi_k \cdot pdf_{ki}(\pi_k), d\pi_k
\]

So the mean availability $A_k(t)$ of the stochastic component $k$ can be calculated analytically by summing the mean probabilities $p_{ki}(t)$ of all states obtained in (4) unless the total failure state 1:

\[
A_k(t) = \sum_{i=1}^{\pi_n} p_{ki}(t)
\]

3. Analytic Uncertainty Propagation in a Stochastic Repairable Multi-states System

Let $S$ be a stochastic non repairable multi-states system composed of $k$ components $\{C_1, C_2, ..., C_k\}$ with $n$ level of performances $\{g_1, g_2, ..., g_{n-1}, g_n\}$. Let $UGF_1$, $UGF_2$, ..., $UGF_{k-1}$, $UGF_k$ be the respective universal generating functions of components $C_1$, $C_2$, ...
By applying the composition operator [13] on UGF of two components \( \Omega (UGF_1, UGF_2) \), the UGF of the resulting sub-system formed by these two components will be: 

\[
UGF_{12} = \Omega (UGF_1, UGF_2); 
\]

by applying again the composition operator on the \( UGF_{12} \) of the sub-system and \( UGF_j \) of the third component \( C_j \), \( UGF_{123} = \Omega (UGF_{12}, UGF_j) \) is obtained. This process is repeated recursively to obtain the UGF of the whole system \( (UGF_s) \). The states having the same level of performance are grouped in a one state that has a probability equal to the sum of probabilities to be in each of these states. Doing this procedure \( (UGF \) technique) for all resulting states of the system, the number of states of the system is reduced and the model is simplified [6]. The resulting UGF of the system \( (UGF_s) \) is function of all UGF, of the components and so it is function of all failure rates of all the components of the system. By introducing uncertainty on each failure rate, these uncertainties will be integrated in the \( UGF_s \). Consequently, the probabilities of the system states are function of probabilities of states of different components. Then, the system state probabilities are calculated independently by knowing the probability of component states on which this system state is depending on. Taking account the uncertainty on failure rate, the probability of component states can be calculated as mentioned in (4).

4. Analytic Uncertainty Propagation in a Stochastic Repairable Multi-states System

In this section, we will explain how the method can be applied to calculate analytically the probability of system states and the availability of the system. Let S a stochastic system composed of three components \( C_1 \), \( C_2 \) and \( C_3 \) where \( C_1 \) and \( C_2 \) are connected in a parallel form and \( C_3 \) in series with the others (See [7] for details). Let \( C_j \) be a repairable component having three states of performance: \( g_{31}, g_{32} \) and \( g_{33} \) with respective probabilities \( p_{31} \), \( p_{32} \) and \( p_{33} \). In the same way, let \( (p_{11}, p_{12}) \), \( (g_{11}, g_{12}) \) and \( (p_{21}, p_{22}) \) be the corresponding states’ probabilities and states’ performances of components \( C_1 \) and \( C_2 \) respectively. Let \( UGF_1 \), \( UGF_2 \) and \( UGF_3 \) be the universal generating functions that model repairable Markov process of components \( C_1 \), \( C_2 \) and \( C_3 \) respectively. By applying the composition operator, the UGF for the parallel block \( (C_1 \) and \( C_2 \) is obtained: 

\[
UGF_{12} = \Omega (UGF_1, UGF_2) = P_{12}(t)Z^e + P_{12}(t)Z^{1.5} + P_{12}(t)Z^2 + P_{12}(t)Z^3 \quad (8)
\]

By applying again the composition operator and using UGF technique to regroup states having same performance, a system with 5 states of performance is obtained:

\[
UGF_s = \Omega (UGF_{12}, UGF_3) = \sum_{i=1}^{5} p_i(t)Z^i \quad (9)
\]

\[
\begin{align*}
g_1 &= 0, \quad p_{11}(t) = p_{11}(t) p_{21}(t) + p_{11}(t) p_{22}(t) + p_{11}(t) p_{23}(t) + p_{12}(t) p_{23}(t) \\
g_2 &= 1.5 \ t/min, \quad p_{21}(t) = p_{21}(t) p_{22}(t) [p_{22}(t) + p_{23}(t)] \\
g_3 &= 1.8 \ t/min, \quad p_{31}(t) = p_{31}(t) p_{22}(t) \\
g_4 &= 2.0 \ t/min, \quad p_{32}(t) = p_{32}(t) p_{23}(t) \\
g_5 &= 3.5 \ t/min, \quad p_{33}(t) = p_{33}(t) p_{23}(t)
\end{align*}
\]

Then, the availability of the system \( A_s(t) \) can be obtained as:

\[
A_s = \sum_{i=2}^{5} p_i(t) \quad (11)
\]

As for a stochastic component, the states of a stochastic system are disjoined and so the probability to be in each state \( p_i(t) \) can be calculated separately. Considering uncertainty on failure rates, the different probabilities \( p_{ij} \) of components \( C_i \) to be in state \( i \), (formulas (10)) can be calculated separately as mentioned in § 2 by using the formula (6).

5. Case study

5.1 Binary State Non Repairable Component with Markov Process

Analytical Uncertainty Propagation for Availability Assessment of Stochastic Multi-state Systems
Let consider $C_2$ as 2 states non repairable component so $UGF_{z}(z)=P_{21}(t)Z^{0}+P_{22}(t)Z^{0}$ and $g_{22}=2.0$ and $g_{21}=0$. For calculus simplification, we consider in this case that repairable rate $\mu_{21}=10$/year with a standard deviation $\sigma$, then an uncertainty according to the normal law $N_{21}(10,\sigma)$. As explained in § 2, we have $dS_2$: (1): $dp_{21}/dt=-\lambda_{21}P_{22}$ and (2): $dp_{22}/dt=\lambda_{22}P_{23}-\lambda_{23}P_{22}$.

The solution of this system $S_2$ is: $P_{21}=-\exp(-\lambda_{21}t)+1$ and $P_{22}=\exp(-\lambda_{21}t)$.

The Jacobi matrix $J_2=-t\exp(-\lambda_{21}t)$. The determinant of $J_2$ is $\det J_2=-tP_{22}$.

The Jacobi matrix is:

$$
\begin{pmatrix}
\frac{1}{\sigma}\sqrt{2\pi}e^{-\frac{(x-m)^2}{2\sigma^2}} \\
\frac{1}{\sigma}\sqrt{2\pi}e^{-\frac{(x-m)^2}{2\sigma^2}} \\
\frac{1}{\sigma}\sqrt{2\pi}e^{-\frac{(x-m)^2}{2\sigma^2}} \\
\end{pmatrix}
$$

(12)

5.2 Three States Non Repairable Component with Markov Process

Let consider $C_3$ as 3 states non repairable component, with its corresponding UGF, $UGF_{z}(z)=P_{31}(t)Z^{0}+P_{32}(t)Z^{0}+P_{33}(t)Z^{0}$, where $g_{32}=4.0$, $g_{31}=1.8$ and $g_{33}=0$. For calculus simplification, we consider in this case that repairable rates $\mu_{31}=\mu_{32}=0$. The failure rates are considered independent of time and have an uncertainty modelled by Normal probability functions. The failure rate means are: $\lambda_{31}(p_{32})=h_{3}(p_{32})=\log(p_{32})/t$. From § 2 and by applying formulas (1), (3) and (4), we have:

$$
\begin{align*}
\text{pdf}_{21}(x) &= \frac{1}{\sigma}\sqrt{2\pi}e^{-\frac{(x-m)^2}{2\sigma^2}} \\
\text{pdf}_{22}(p_{22}) &= \text{pdf}_{21}(h_{3}(p_{22})), (1/\det J_2(h_{3}(p_{22})))^{-1} = \left(\frac{1}{\sigma}\sqrt{2\pi}e^{-\frac{(p_{22}-m)^2}{2\sigma^2}}\right)^{1/\det J_2(h_{3}(p_{22}))} \\
\det J_2(h_{3}(p_{22})) &= -1.
\end{align*}
$$

Let consider $S_3$ a system of 3 equations where $\lambda_{31}$ and $\lambda_{32}$ are the unknown variables and are independent of time “t” as considered above. The solution $SO_3$ of this system has the form (where $W$ is the Lambert’s function):

$$
\lambda_{31}(p_{32},p_{33}) = h_3(p_{32},p_{33}) = \frac{-\log(p_{32})}{t}.
$$
Analytical Uncertainty Propagation for Availability Assessment of Stochastic Multi-state Systems

We have pdf, p(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.

Let det(J\|h_2(p_{32},p_{31}), h_1(p_{22},p_{31})) be the determinant on P_{32} and P_{31}. The states of C_3 are independent, then pdf_{32,31} = pdf_{32} \cdot pdf_{31}. So:

pdf_{P_2}(p_{32}) = pdf_{P_2}(h_2(p_{32},p_{31})) \cdot pdf_{P_2}(h_1(p_{22},p_{31})), det(J\|h_2(p_{32},p_{31}), h_1(p_{22},p_{31}))^{-1}

6. Availability Assessment: Numeric Example

6.1 Example of a Non Repairable Component

As in § 5.1, let consider C_2 as 2 states non repairable component where failure rate \( \lambda_2 = 10/\text{year} \). Let consider two cases: without uncertainty, and with uncertainty on failure rate. Without uncertainty, and from § 5.1, we have \( p_{21} = e^{-\lambda_2 t_1} + 1 \) and \( p_{22} = e^{-\lambda_2 t_1} \), with \( p_{22} \) the probability that the C_2 is in its nominal state, so the availability is \( A(t) = e^{-\lambda_2 t_1} \). The uncertainty on failure rate \( \lambda_2 \) is modeled by a Normal probability law with the mean \( m = 10 \) and standard deviation \( \sigma = 2 \): \( N(10, 2) \). Then, the failure rate uncertainty pdf is:

\[
pdf_{P_2}(p_{22}) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\log p_{22} - m)^2}{2\sigma^2}}.
\]

With the formula (6) we obtain the corresponding mean value of the probability \( p_{22}(t) \):

\[
\frac{\pi_{22}}{\pi_{22}}pdf_{P_2}(\pi_{22})d\pi_{22} = -\frac{1}{2} e^{-\frac{(1-2m+2\sigma^2)}{2}} \left[ \text{erf} \left( \frac{-(m+t+\sigma^2)}{\sqrt{2\sigma}} \right) - 1 \right]
\]

So, the mean value of the availability when having uncertainty on failure rate is:

\[
A_{\text{uncer}} = -\frac{1}{2} \left[ \text{erf} \left( \frac{-(m+t+\sigma^2)}{\sqrt{2\sigma}} \right) - 1 \right]
\]

This expression needs to be normalized dividing its by \( \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{m}{\sqrt{2\sigma}} \right) \), but this coefficient is near to be equal to 1 when \( \sigma << m \), then it is neglected here. The differences observed in the Table 1 are not so important and mean values of availability (with uncertainty) are greater than the exact value of the availability. Then, it seems that the availability assessment without uncertainty is a little bit conservative.

| t  | \( A \) | \( A_{\text{uncer}} \) | \( |A - A_{\text{uncer}}| \) | \( \text{uncer}(MC) \) | \( |A - \text{uncer}(MC)| \) |
|----|-------|----------------|-----------------|----------------|----------------|
| 0.04 | 0.6703 | 0.6725 | 0.0022 | 0.3051 | 0.0041 |
| 0.1  | 0.3679 | 0.3753 | 0.0074 | 0.6614 | 0.0041 |
| 0.16 | 0.2019 | 0.2125 | 0.0106 | 0.8491 | 0.0042 |

Also, one remarks that having uncertainty on the mean of failure rate may increase the error on the availability estimation when component is supposed in good performance...
phase (beginning of working life, time < 0.1 year). In a second step, it is proposed to measure the sensitivity to changes in the uncertainty characteristics (parameters of the law and/or change of uncertainty law). For example, it is possible to consider now that the uncertainty on failure rate $\lambda_{21}$ is modeled by a Normal probability law with the mean $m$ and with fixed $\sigma = 2: N (m, 2)$. By using the formula of $A_{\text{uncer}}$ given above and by doing $10^5$ Monte Carlo calculations of $A_{\text{uncer}}$ where $m$ is taken randomly and uniformly in the interval $[9, 11]$, we obtain histograms shown below (Fig. 2).

![Histograms of Availability with Uncertain $\lambda$ when $t=0.04$ year (left) and $t=0.16$ year (right)](image)

The histograms in Fig. 2 show that the uncertainty on availability means is dependent on the time; the change of the slope indicates that the lower availability mean values are more and more important if a family of uncertainty pdf is studied (here normal laws). The corresponding mean values are closed to the analytical ones (Table 1) with uncertainty, which validates the quality of the convergence of Monte Carlo simulation: mean(0.04) = 0.6726, mean(0.1) = 0.3758, mean(0.16) = 0.2134.

So, because of these histogram slopes, the effect of uncertainties with the time could be equilibrated. The difference between exact availability and the availability mean value is greater for large times but the uncertainty to the knowledge on the mean failure rate has more effect for lower mean values for large times.

These aspects are very important for decision making process because of the possible overestimation given by the unique availability assessment and clearly, the non negligible effects of uncertainties (the probability that availability is lower than its assessment without uncertainty is not negligible).

### 6.2 Example of a Multi-state System

The uncertainty propagation is taken into account for the availability assessment of the multi-states system proposed in § 4.1. The unavailability is equal to (see formula 10, 11):

$$p_{11}(t) p_{22}(t) + p_{21}(t) p_{12}(t) + p_{31}(t) p_{11}(t) p_{22}(t)$$

To simplify the calculations, the 3rd component is supposed to have two states as the others. The resulting pdf associated to the unavailability can be built by integration, but it is more convenient here to calculate this by Monte Carlo simulation. It is supposed that the uncertainty laws are normal for each failure rates each state of the components. Then, using the formula (13), it is possible to generate the random values for $p_{12}$, $p_{22}$ and $p_{12}$ which are respectively $1-p_{11}$, $1-p_{22}$ and $1-p_{31}$. If necessary, it is possible to consider the law of $g(Y)$ for $Y = 1- X$, with the pdf $f(X)$ for $X$. Its expression is easily obtained knowing $f$: $g(Y) = f(1-Y)/(1-Y)$ when $0 < Y < 1$. This last relation is useful for example to deduce the availability pdf knowing the unavailability pdf.
Then, the unavailability is obtained with a simple Monte Carlo simulation based on its expression above and the uncertainty pdf given in formula (13). An inversion procedure can be used to generate the random probabilities $p$ according to this pdf based on the generation of random numbers $\xi$ uniformly distributed in $[0,1]$ (erfinv is the reciprocal function of error function erf):

$$p = \exp\left[\sqrt{\frac{2}{\pi}} \text{erfinv}\left(2\xi - 1\right) - m\right].$$

The parameters for each failure rates pdf are ($i$: component, $j$: state): $m_{i1} = 7$, $\sigma_{i1} = 1$; $m_{i2} = 10$, $\sigma_{i2} = 2$; $m_{i3} = 7$, $\sigma_{i3} = 1$. The figure 3 show the unavailability pdf histograms for $t = 0.04$ and $0.16$ /year. It is interesting to see the uncertainty effect of combined failure rates to the unavailability with its pdf (and consequently to the availability). The pdf shape becomes more and more asymmetric when $t$ increases.

![Figure 3: Histograms of Unavailability of a MSS with Uncertain $\lambda$ when $t=0.04$ year (left) and 0.16 year (right)](image)

The mean values in Table 1 (right) show that the unavailability assessment without uncertainty is more conservative when time increase, the mean values obtained with uncertainty being more and more lower than without uncertainty. This effect is confirmed by the asymmetry observed in Fig. 3. Nevertheless, this situation seems stable for large times and the deviations around the mean stay important for decision making. This calculation could be proposed for the different states of the studied MSS and in the case of more number of states for each component.

7. Conclusion

In this paper a method of analytic propagation of uncertainty on failure rate adapted to UGF method in stochastic multi-states system is proposed. This method covers many types of process and large scale systems to assess the availability. Also, it treats uncertainty on failure rate based on usual probability laws (uniform, normal, or exponential laws) which corresponds some possible physical phenomena. Additionally, other uncertainty models and especially on other input data (repairable rate, performance levels, etc.) can be propagated by this method. A propagation of uncertainty on failure rates of all components in the system using analytic evaluation of probability to be in each state of the MSS, and consequently its availability and associated pdf. This method can reduce the computing time in a case of calculus with Monte Carlo simulation and, for a given computing time, improve the availability uncertainty assessment.

For two-state and three-state stochastic non repairable component, we detailed the calculus of probability density function of each state and the formulas have been evaluted. These formulas can be injected in the (9) equations to deduce the probabilities of the given system in paragraph 3 and to assess analytically its availability. The unavailability and its uncertainty (pdf) have been studied for a simplified MSS (the
specific state for which the performance \( g = 0 \). The corresponding pdf shape has been studied with time and has showed an effect to asymmetry distributions. In all these cases, the availability (and unavailability) without uncertainty have been compared with the mean value with uncertainty. It shows a favorable difference proving a conservative behavior of the assessment without uncertainty, but a noticeable deviation around the mean value when uncertainty is taken into account. The relative effects according to parameters as the standard deviation could be studied in more detailed way. More generally, the dependence to the parameters of the uncertainty law should be improved with numerical advanced tests in future works. The corresponding results could be improved by introducing global sensitivity analysis and Sobol indices, and by comparing to results obtained by Monte Carlo simulation [11] [12]. Also, other tests needs to be developed to prove the time computing gain with so approaches compared to Monte Carlo simulation and to improve the demonstration of the advantages for decision making in MSS availability assessments.

Another objective after this work is to do analytical calculus for repairable multi-states system and to link this kind of research with reliability optimization problems or maintenance planning optimization problems with uncertain data. Finally, it can be noticed that the Bayesian approach for data uncertainty assessment can be included in the proposed approach in terms of pdf determination (and adaptation) of inputs. Then, the effect of this approach to the availability assessment should be evaluated.

References


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