Load-strength Dynamic Interaction Principle and Failure Rate Model

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Abstract: In majority of engineering structures and mechanical components/systems, the operating load is usually varying randomly. The material property also suffers degradation with the loading conditions. This leads to continuously changing load-strength relationship and results in a variation in the failure rate. In the load amplitude domain, the situation can be described as one in which the structure or component is subjected to multiple actions of stochastic load, and the material property degrades continuously with loading pattern. Since a failure is caused when the load applied becomes higher than the relevant strength, a closed-form failure rate model is developed for a mechanical component subjected to a random load process. Consequently, the variation in failure rate is interpreted in both the load uncertainty and the component property uncertainty. Based on such a failure rate model, the effects of component strength degradation, as well as load dispersion and component strength dispersion on the failure rate are highlighted. The present paper analyzes failure rate evolvement from the competing mechanisms of statistically-increasing load and degrading strength and interprets the three stages uniformly as the manifestation of ever-changing load-strength interaction.

Keywords: Failure rate, load uncertainty, strength uncertainty, strength degradation, load-strength interaction

1. Introduction

Failure rate is an important measure of reliability. For most mechanical components or structures, failure rate changes considerably during service life. Generally, the change depends on product type, load characteristics, failure mechanism and other operational profiles [1].

The typical failure rate curves take on a bathtub shape (Fig. 1), comprising three types of failure rate versus time relationship in a sequence of decreasing, constant (approximately) then increasing failure rate. The three stages for a failure rate curve are conventionally interpreted as the infant mortality phase (stage I), the chance failure phase (stage II), and the wear out phase (stage III). These three stages, are traditionally interpreted as material and/or manufacturing defect dominated, random load induced, and material deterioration determined, respectively.
It is thought that the mortality phase demonstrates a subpopulation dominated by quality-control defects due to poor workmanship, contamination, out-of-specification parts and materials, and other substandard manufacturing practices. The other two phases are attributed to stochastic load and product performance deterioration, respectively.

![Typical failure rate curve (bathtub-shaped curve)](image)

**Figure 1:** Typical failure rate curve (bathtub-shaped curve)

Theoretically, the failure rate function can be expressed as the ratio of the life probability density function $f(t)$ to the reliability function $R(t)$, i.e. $h(t) = f(t) / R(t)$. Meanwhile, the traditional reliability calculation is commonly carried out through the failure rate function as $R(t) = \exp\left\{ -\int_0^t h(t) \, dt \right\}$ [2]. For mechanical components or structural parts, it is well known that reliability can be easily expressed through the load-strength interference relationship [3]. However, to date no equation has been developed to bridge the gap between the failure rate and the load-strength interference relationship. Owing to the absence of a physical principle based failure rate model to link the failure rate with load/strength parameters, the understanding of the characteristics of the failure rate curve is still phenomenological, and the pertinent interpretation is partial and qualitative.

Although none of the most widely used life distributions, such as normal distribution, log-normal distributions and Weibull distribution etc., will induce the three-stage failure rate curve, several conditions for unimodal mean residual life were developed to imply that the failure rate function has a bathtub shape [4]. In terms of failure rate and distribution classes, Bae et al [5] investigated the link between a practitioner’s selected degradation model and the resulting lifetime model, showing that the seemingly innocuous assumptions of the degradation path create surprising restrictions on the lifetime distribution. It is also thought that mechanical systems may not appear to have an infant mortality period [6].

Failure rate is usually discussed with reference to life distribution [7-9]. As to the failure mechanisms typical for mechanical components or structures, such as deformation, fracture under static load or fatigue under cyclic load, the number of load applications is a more relevant parameter than calendar time. For instance, taking into account the effect of a stochastically repeated load, a load-number-dependent reliability formula was proposed with the number of load applications, $n$, as an explicit parameter [10]:

$$R(t) = \exp\left\{ -\int_0^t h(t) \, dt \right\}$$
\[ R(n) = \int_0^\infty f(x)[\int_0^x g(y)dy]^n dx \]

where, \( R(n) \) denotes the component reliability after \( n \) load applications, \( f(x) \) denotes the component strength probability density function, and \( g(y) \) the stress probability density function.

Taking the view that the failure rate function can be perfectly characterized through the load-strength interference relationship, and that the product quality shortcoming is already reflected by strength distribution, it will be shown that, among other things, the failure rate decline in the early part of a product’s service life is also determined by strength distribution and load distribution. It is evident that, if a product has successfully operated for a period of time, and product strength does not degrade during its service life, the product will never fail under a load that is less than those it has previously successfully resisted. Meanwhile, the likelihood that a higher load appears will become less and less with the increase in loading experience. When the component property deterioration is taken into account, it is reasonable to expect that different strength degradation modes will yield different failure rate patterns.

2. Order Statistics and Failure Rate Model

Consider a set of i.i.d. (independent, identically distributed) load variables. Let \( y_1, y_2, \ldots, y_n \) denote \( n \) load samples, and \( Y_{(k)} \) \((k=1\sim n)\), the \( k \)-order statistic in a sample of size \( n \), denote the \( k \)-th smallest of the \( n \) samples in statistical sense. With the probability density function \( g(y) \) and the cumulative distribution function \( G(y) \) of the load random variable \( y \), the probability density function of the 1-st order statistic \( Y_{(1)} \) (denoted by \( g_{(1)}(y) \)) and that of the \( n \)-th order statistic \( Y_{(n)} \) (denoted by \( g_{(n)}(y) \)) are, respectively [11]:

\[ g_{(1)}(y) = n[1 - G(y)]^{n-1} g(y) \]  
\[ g_{(n)}(y) = n[G(y)]^{n-1} g(y) \]

Fig.2 illustrates the relationship between a random load distribution and the maximum load order statistic distributions for sample sizes of 10, 20, 50, and 500, respectively.

![Figure 2: Distributions of a load r.v. and its maximum order statistics](image)

(Legend: o.s. → order statistic)

By incorporating the load (stress) order statistic into the conventional load (stress) - strength interference model, the dynamic (time or load experience dependent) component failure probability model and the failure rate model can be developed. The dynamic
characteristic of such a model comes from the ever changing maximum order statistic distribution. That is, the distribution of the maximum load order statistic will vary continuously with the increase of load experience. For example, the component failure probability after \( n \) load actions can be modeled as (see Fig.3)

\[
P(n) = \int_0^\infty f(x) \int_x^\infty g_s(y) dy \, dx
\]  

(3)

\[f_S(n)(x)\] denotes the strength distribution of the surviving samples after \((n-1)\) load applications.

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**Figure 3:** Load order statistic-strength interference relationship

To develop the component failure rate model, we begin with a general load-strength interference relationship (shown in Fig.4) from which component failure probability subjected to a single load action can be expressed as

\[
P = \int_0^\infty f(x) \int_x^\infty g(y) dy \, dx
\]  

(4)

\[f_S(n)(x)\] (shown in Fig.5). That is, based on the strength distribution of the surviving samples, the failure rate can be expressed as

\[
h(n) = \int_0^\infty f_S(n)(x) \int_x^\infty g(y) dy \, dx
\]  

(5)

where, \(f_S(n)(x)\) denotes the strength distribution of the surviving samples after \((n-1)\) load applications.

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**Figure 4:** Load-strength interference relationship

By definition, the failure rate at time \( t \) or at the \( n \)-th load action, denoted by \( h(t) \) and \( h(n) \) respectively, is the probability of failure, given survival to time \( t \) or cumulative load number \((n-1)\). To calculate failure rate \( h(n) \) by means of the load-strength interference relationship, it is necessary to know the strength distribution of the surviving samples after \((n-1)\) load applications, \(f_S(n)(x)\) (shown in Fig.5). That is, based on the strength distribution of the surviving samples, the failure rate can be expressed as

\[
h(n) = \int_0^\infty f_S(n)(x) \int_x^\infty g(y) dy \, dx
\]  

(5)
To satisfy the condition of $\int_{0}^{\infty} f_{n(n)}(x)dx = 1$, the strength distribution of the samples surviving $(n-1)$ load actions can be approximately expressed, by modifying the original strength distribution, as

$$f_{n(n)}(x) = \frac{f(x)}{\int_{z(n)}^{\infty} f(x)dx} \quad (z < x < \infty)$$

(6)

where, $z(n)$ denotes the maximum of the $(n-1)$ times the load.

Since the maximum load $z$ is also a random variable following the maximum order statistic distribution, then according to the principle of calculating the expectation of a random function, the following basic failure rate equation can be developed (ref. Fig.6).

For the sake of simplification, $z(n)$ is also expressed simply as $z$ in the following)

$$h(n) = \int_{0}^{\infty} g_{n-1}(z) \left[ \int_{z}^{\infty} f(x)dx \left[ \int_{0}^{\infty} g(y)dy \right] dx \right] dz$$

(7)

or

$$h(n) = \int_{0}^{\infty} g_{n-1}(z) \left[ \int_{z}^{\infty} f(x)dx \left[ \int_{0}^{\infty} \frac{f(x)}{f(x)dx} dy \right] \right] dz$$

(8)
3. Effect of load/strength Dispersion on the Failure Rate at the Mortality and Chance Failure Stages

In order to demonstrate the effect of load distribution and component strength distribution on failure rate, several load-strength combinations are considered below. Four different load-strength interference relationships are illustrated, with the loads and strengths assumed to follow normal distributions. The respective expectations and standard deviations are listed in Tab.1. Where, \( \mu_y \) stands for the mean of stress, \( \sigma_y \) stands for the standard deviation of stress; \( \mu_x \) stands for the mean of strength, and \( \sigma_x \) stands for the standard deviation of strength. The failure rate curves corresponding to the different load-strength combinations are labeled as “base line”, “high load std”, “high strength std” and “higher load and strength std”, respectively (shown in Fig.7). It is demonstrated that load-strength distribution characteristics have considerable effect on failure rate, and it can be evaluated by the proposed model (Eq.7 or Eq.8).

### Table 1: Parameters for the different load/strength distributions

<table>
<thead>
<tr>
<th></th>
<th>( \mu_y )</th>
<th>( \sigma_y )</th>
<th>( \mu_x )</th>
<th>( \sigma_x )</th>
<th>( \sigma_y/\sigma_x)_{\text{std}}^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base line</td>
<td>400</td>
<td>40</td>
<td>600</td>
<td>40</td>
<td>0.707</td>
</tr>
<tr>
<td>High load std</td>
<td>400</td>
<td>80</td>
<td>600</td>
<td>40</td>
<td>0.894</td>
</tr>
<tr>
<td>High strength std</td>
<td>400</td>
<td>40</td>
<td>600</td>
<td>80</td>
<td>0.447</td>
</tr>
<tr>
<td>Higher load/strength std</td>
<td>400</td>
<td>60</td>
<td>600</td>
<td>60</td>
<td>0.707</td>
</tr>
</tbody>
</table>

![Figure 7: Failure rate curves corresponding to different load/strength combinations](image)

4. Effect of strength degradation on failure rate

Under the durative action of cyclic loading, or the effect of other environment factors such as corrosion or aging, material property will degrade, and residual strength will decrease gradually. First, consider the situation that material strength degrades exponentially as described by Eq.9 and shown in Fig.8.

\[
S(n) = S_0\left(1 - \left(\frac{n}{N}\right)^\gamma\right)
\]

(9)

where, \( S(n) \) stands for the material strength after \( n \) load applications, \( n \) stands for the number of load cycles, \( S_0 \) stands for the original material strength, \( N \) stands for the...
baseline number of load cycles, e stands for the strength degradation exponential characterizing the strength degradation trend.

![Material strength degradation curves](image)

**Figure 8:** Material strength degradation curves

With material strength degradation incorporated, the basic failure rate equation (Eq.8) becomes:

\[
h(n) = \int_0^\infty g_{n-1}(z) \left\{ \int_z^\infty g(y) \left[ \int_y^\infty f_{x}(x,n) \frac{dx}{\int f_{x}(x,n)dx} \right] dy \right\} dz
\]

(10)

where, \( f_{x}(x,n) \) is the pdf (probability density function) of the residual strength after the stress is applied \( n \) times.

For the normally-distributed strength \( f(x,n) \sim N(\mu_x(n), \sigma_x) \), the mean value will vary with the number of load actions, \( n \) as

\[
\mu_x(n) = S_0 \left( 1 - \left( \frac{n}{N} \right)^e \right)
\]

(11)

In the condition where no more strength degradation data available, the strength standard deviation \( \sigma_x \) can be assumed constant, i.e.

\[
\sigma_x(n) = \text{constant}
\]

(12)

The typical failure rate curves corresponding to the different load-strength combinations and different strength degradation exponentials are shown in Fig.9. It demonstrates that different strength degradation trends yield obviously different failure rate curves.

When the material strength degrades logarithmically as a function of life fraction \( n/N \), under the action of cyclic stress, the residual strength, \( S(n) \), can be expressed

\[
S(n) = A + B \ln(1 - n/N)
\]

(13)

where, \( n \)-number of the applied cyclic stress cycles, \( N \)-number of cycles to failure under the applied cyclic stress, \( A, B \)-coefficients.
Since $n = 0$ refers to the initial material state and the corresponding strength equals the ultimate tensile stress $S_0$, so $A = S_0$ in Eq.13. When $n = N - 1$, the residual strength will reduce to its critical value, i.e. $S = \sigma_{\text{max}}$, where $\sigma_{\text{max}}$ is the maximum value of the applied cyclic stress. Thus, it is known that $B = (S - \sigma_{\text{max}}) / \ln N$, and Eq.13 becomes:

$$S = S_0 + (S_0 - \sigma_{\text{max}}) \ln(1 - n / N) / \ln N$$

(14)

Fig.10 shows the test data of the normalized carbon steel and the logarithmic strength degradation model, in which the ultimate tensile strength is 1180MPa (the true strength is based on the actual residual area after fracture and the applied load at the moment of fracture).

**Figure 9**: Failure rate curves

(a) $\mu_y = 400, \sigma_y = 40; \mu_x = 600, \sigma_x = 60$

(b) $\mu_y = 450, \sigma_y = 20; \mu_x = 600, \sigma_x = 60$

**Figure 10**: Strength degradation test data and model

When the logarithmic material strength degradation equation is incorporated into the basic failure rate model, the failure rate curves take on the typical bathtub curve shape as shown in Fig. 11, where different curves correspond to different load distribution - strength distribution combinations. Comparing the failure rate curves obtained in the case of
exponential strength degradation, it can be observed that different strength degradation modes yield considerably different types of failure rate curves.

![Typical bathtub-shaped failure rate curve](image1)

![Failure rate curves corresponding to different load/strength distributions](image2)

**Figure 11:** Failure rate curve based on logarithmic strength degradation

### 5. Conclusion

A failure rate model has been developed based on the dynamic load-strength competing mechanism. Different load/strength distribution combinations and different strength degradation modes were considered, with the corresponding failure rate versus the number of load action relationships presented. The results highlighted the considerable effect of the statistical characteristics of load and strength on the shape of the failure rate curve and also the significant role of strength degradation mode. If component strength does not degrade, the failure rate curve take on the features of the first two stages of a typical three-stage bathtub-shaped curve, i.e., the failure rate decreases continuously with operation time or the number of load cycles experienced, with lower and lower descending trend respectively. When component strength degradation is taken into account, it was clearly illustrated that the failure rate first decreases and then increases with operation time or load action number, and the failure rate curve takes on the typical bathtub shape.

For a typical bathtub-shaped failure curve, the quick decline of the failure rate in the first stage can not merely be attributed to defects within the product. The relationship between failure rate and the action number of a stochastic load demonstrates that the decline in the failure rate curve is determined by both the strength distribution and the load distribution. In the same way, the chance failure stage and wear out stage are characterized by the ever changing load-strength relationship.

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**References**


