Robust Design using Variance Upper Bound Theorem

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Abstract: The exact upper bound of the variance of properties from multiple sources is attained from sampling not more than two sources. This paper discusses important applications of this result referred to as variance upper bound theorem. A new conservative, non-parametric estimate has been proposed for the capability index of a process whose output combines contributions from multiple sources of variation. A new method for assessing and increasing the robustness of processes, operations and products where the mean value can be easily adjusted or is not critical has been presented, based on the variance upper bound theorem. We show that the worst-case variation of a property from multiple sources, obtained by using the variance upper bound theorem, can be used as a basis for developing robust engineering designs and products. If a design is capable of accommodating the worst-case variation of the reliability-critical parameters, it will also be capable of accommodating the variation of the reliability-critical parameters from any combination of sources of variation and mixing proportions. In this respect, a new algorithm for virtual testing based on the variance upper bound theorem has been proposed for determining the probability of a faulty assembly from multiple sources.

For sources of variation that can be removed, the robustness can be improved further, by removing the source that yields the largest decrease in the variance upper bound. Consequently, the correspondent algorithm is also presented. A number of engineering applications have been discussed where the variance upper bound theorem can be used to assess and increase the robustness of mechanical and electrical components, manufacturing processes and operations.

Keywords: Variance upper bound theorem, virtual testing, variability, reliability, robustness

1. Introduction

Variability associated with critical design parameters (e.g., material properties and dimensions) is a major source of unreliability. Material properties such as yield stress, fracture toughness, modulus of elasticity, density, resistivity, etc., often appear as reliability-critical parameters. Most of the failures, especially the ones occurring early in life are quality failures due to the presence of substandard items that find their way into the end products. Production variability during manufacturing, not guaranteeing the specified tolerances or introducing flaws in the manufactured product, promote early-life failures. Reducing these can be achieved by a statistical process control based on monitoring the variations of the output parameters [1]. Another powerful tool for reducing variability is the six-sigma quality philosophy [2] based on production with very small number of
defective items. A comprehensive discussion related to the effect of variability on the reliability of products can be found in [3-6]. Depending on the source, the same component of the same material manufactured to the same specification is usually characterized by different properties. Between sources, variation exists even if the variations of property values characterizing the individual sources are very small. Furthermore, due to the inherent variability of the manufacturing processes, even items from the same manufacturer are usually characterized by different properties. Because of the natural variation of critical design parameters, early-life failures are often due to unfavourable combinations of values (e.g., worst case tolerance stacks) rather than due to a single production defect.

More robust processes and designs can be achieved by identifying the sources of variation whose removal yields the largest decrease in the maximum variation of the property for any possible mixing proportions from the sources. This defines the focus of the present paper.

It is important to emphasize, that the subject of this paper is not the probability that a parameter will exceed or fall below a particular critical limit. In most of the considered examples, only the variances characterizing the sources of variation are known. The underlying distributions from the sources are usually unknown and probability of exceeding or falling below a particular value cannot be calculated. Furthermore, the mixing proportions with which the sources of variation are present are usually unknown.

Despite the significant amount of uncertainty surrounding the underlying distributions and their mixing proportions, the variance upper bound theorem (VUBT) first formulated and proved in earlier work [7] helps to determine the exact upper bound of the variation from multiple sources. In the same work, an application of the VUBT for determining the uncertainty associated with the fracture toughness at a specified test temperature of steels has been discussed. It is a well established fact that taking samples from an inhomogeneous microstructure results in a large variation of the properties. In this respect, the VUBT is particularly important if a precise conservative estimate of the uncertainty associated with the properties from the sources is needed. In this case, the sources of variation are the micro structural zones and the mixing proportions are the probabilities with which these micro structural zones are sampled (Fig.1). In the common case where the probabilities with which the micro structural zones are sampled are unknown, the worst possible scatter of properties from arbitrary sampling of the micro structural zones can be obtained on the basis of the VUBT.

The present paper continues the development of the work initiated in [7] by applying the VUBT, as a basis for a new worst-case design method aiming at improving the robustness of processes, operations and products originating from multiple sources for which the worst performance is associated with the worst variation of a property around the common mean.

For these type of processes or products, the common mean can be easily adjusted to a specified target value or is not critical, but deviations of the parameter from the common mean lead to undesirable performance. It is a well-known fact that while the mean value of the output of a process can often be easily altered on a specified target value, the variance cannot be altered so easily. Reducing the variance of a process usually requires fundamental technological changes which need a substantial investment. In this paper, robustness will denote the capability of a process or a product to cope with variability, with minimal loss of functionality.
The emphasis in this paper is in line with the Taguchi on-target engineering philosophy [6] and with the fundamental components of quality defined by Juran [8]: (i) the product features and (ii) the product’s conformance to those features. Product or process conformance to a required target means that quality is improved when the maximum variation of the output is minimized.

This powerful approach for delivering robust designs is illustrated in Fig.2. Product (process) B is more robust and performs better than product (process) A, because the key output parameter characterizing product B is more often close to the target (optimum) value than product A. Conformance to a customer-defined target also means that quality improves when variation in performance is minimized [6].

The selected approach is also in line with the worst-case design philosophy [9] and the worst-case philosophy of the classical decision theory [10].

2. Variance upper bound theorem: an overview

Distribution mixtures [11] find a wide application in modeling the variation of the properties from multiple sources (Fig.3).

In earlier work [12], we demonstrated that the distribution of properties from a heterogeneous microstructure including several micro structural zones can be modelled by a mixture of several distributions each of which corresponds to a separate micro structural
zone. As a result, the cumulative distribution function \( F(x) \) of a property from \( M \) sources (micro structural zones) is modelled by the finite mixture distribution

\[
F(x) \equiv P(X \leq x) = \sum_{k=1}^{M} p_k F_k(x)
\]  

(1)

\[
\sum_{k=1}^{M} p_k = 1 \text{, where } F_k(x), \ k = 1, \ldots, M \text{ characterizes the distribution of properties from the separate sources (zones) and } p_k, (k = 1, M) \text{ are the mixing proportions of the sources (the probabilities of sampling the separate zones).}
\]

The mean \( \mu \) of the mixture distribution (1) is given by:

\[
\mu = \sum_{k=1}^{M} p_k \mu_k
\]

(2)

where \( \mu_k \) are the means of the individual distributions composing the mixture [11]. In [7], an expression has been derived for the variance of a finite mixture distribution given by equation (1) as a function only of the pairwise distances between the means \( \mu_k \) of the component distributions:

\[
V = \sum_{i=1}^{M} p_i [V_i + (\mu_i - \mu)^2] = \sum_{i,j} p_i p_j (\mu_i - \mu_j)^2
\]

(3)

Equation (3) is valid for any type of individual distributions characterizing the variation of properties from the sources. According to equation (3), the total variance \( V \) of a property from multiple sources has two major components: (i) a component \( \sum_{k=1}^{M} p_k V_k \) reflecting the variance of the property characterizing the individual sources and (ii) a component \( \sum_{i<j} p_i p_j (\mu_i - \mu_j)^2 \) reflecting the distances between the means of the individual distributions characterizing the separate sources. Both components are scaled by the mixing proportions \( p_i, i=1,M \) (Fig.3).

In the important case where \( p_k \) are unknown, an exact upper limit for the variance of the distribution mixture has been derived in [7]. The main result in this paper is the formulation and proof of the variance upper bound theorem (VUBT) that states: \textit{The maximum variance of properties from sampling multiple sources is always attained from sampling not more than two sources.}

In other words, for items coming from multiple sources, a single source or at most two sources can be found sampling from which produces the maximum variation of properties.

Mathematically, the VUBT can be expressed as

\[
V_{\text{max}} = p_{\text{max}} V_k + (1 - p_{\text{max}}) V_s + p_{\text{max}}(1 - p_{\text{max}})(\mu_k - \mu_s)^2
\]

(4)

where \( k \) and \( s \) are the indices of the sources for which the variance upper bound is obtained and \( 0 \leq p_{\text{max}} \leq 1 \) is the mixing proportion yielding the maximum variance. If \( p_{\text{max}} = 1 \),
the maximum variance is obtained from sampling a single source (the $k$-th source) only.

![Diagram of sources](image)

**Figure 3**: Variation of properties from multiple sources.

The algorithm for finding this upper bound has also been proposed in [7]. Determining the indices $k$ and $s$ of the sources and the sampling probability $p_{\text{max}}$ yielding the upper bound variance from $M$ sources involves only $M(M + 1)/2$ easily performed checks.

The mean $\mu_{\text{max}}$ of the mixture yielding the upper bound variance is obtained from

$$\mu_{\text{max}} = p_{\text{max}} \mu_k + (1 - p_{\text{max}}) \mu_s$$

(5)

3. **A non-parametric and conservative estimate of the capability index**

The VUBT can be used for obtaining a conservative non-parametric estimate of the process capability index when the mixing proportions from the separate sources are unknown. The process capability index is defined as [1]:

$$C_p = \frac{USL - LSL}{6\sigma}$$

(6)

where USL and LSL are the upper and the lower specification limits (Fig.4) and $\sigma^2$ is the variance of the process.

![Normal distribution](image)

**Figure 4**: The mean of a robust process can shift off-centre and the percentage of faulty items can still remain very low.

A large process capability index means that fewer defective or non-conforming units will be produced. A process with a large capability index is a robust process. This means that the process mean can shift off-centre and the percentage of faulty items can still remain very low. If the process is not centered, the actual capability index can be used [1]:
A conservative estimate of the process capability index for properties from multiple sources can be obtained by using an upper bound variance estimate $\sigma^2_{\text{max}}$ produced by the algorithm described in [7]:

$$C_p^* = \frac{\text{USL} - \text{LSL}}{6\sigma_{\text{max}}}$$  \hspace{.5cm} (8)

For a non-centered process, a conservative estimate of the actual capability index can be used

$$C_{pk}^* = \min\left[\frac{\text{USL} - \mu}{3\sigma_{\text{max}}}, \frac{\mu - \text{LSL}}{3\sigma_{\text{max}}}\right]$$  \hspace{.5cm} (9)

For the conservative estimates, the relationships

$$C_p \geq C_p^*; \quad C_{pk} \geq C_{pk}^*$$  \hspace{.5cm} (10)

are valid.

Determining a non-parametric and conservative estimate of the process capability index helps to stabilize the variation of the process within the control limits and reduce the number of faults in the end product. The non-parametric capability index can serve as a basis for ranking, comparing and selecting competing manufacturing processes.

Reducing the number of faults is also very important for the reliability of the product. A latent fault can easily cause failure, if particular triggering conditions are encountered. A material flaw for example, is a fault which may not cause failure during a particular period of operation. However, if the component is overstressed or in corrosive environment or operated for a sufficient length of time so that a fatigue crack has developed, the material fault may lead to a catastrophic failure with heavy consequences. Reducing latent faults also reduces significantly early-life failures that are associated with warranty claims, cost of rework, loss of business and a heavy financial impact due to the high present value of an early failure.

4. A new method for improving robustness of processes, products and operations

The conservative estimate of the variation of properties can be used for developing robust designs and processes, where the mean output can be easily adjusted or is not critical. Indeed, if the design is capable of accommodating without failure the maximum possible variation of the design parameters, it will be capable of accommodating without failure any other variation of the design parameters, produced by any particular combination of mixing proportions from the sources. In other words, the design can be made resistant to the worst possible variations of the design parameters.

If no source of variation can be removed, achieving a robust product/process by using the VUBT can be presented as a three-step process. The first step involves determining the maximum possible variance of the property, by calculating the variance upper bound and the mean $\mu_{\text{max}}$ corresponding to the worst variation given by equation (5). The second step involves adjusting the mean $\mu_{\text{max}}$ of the process with the maximal variance on the target value by adding or subtracting a common value (the difference $t_0$ between the
mean $\mu_{\text{max}}$ of the worst-case distribution mixture and the target value). Adding or subtracting a common value $t_0$ to a distribution alters only its mean and does not alter the variance (Fig.5). Adding or subtracting a common value to all individual distributions does not alter their relative position, it alters only the global mean of the distribution mixture defined on these individual distributions.

In fact, the first two steps are about creating the worst possible variation of the parameter around the target value.

The final step involves verification whether the design can accommodate this worst-case variation. In short, the process of creating a more robust product based on the variance upper bound theorem is a process of making the product resistant against the worst-case variation of the output from the target value (Fig.5a).

![Figure 5: Steps to achieve a robust design](image)

If sources of variation can be removed, achieving a robust design includes: determining the variance upper bound, removing sources that cause a maximum reduction of the variance upper bound and adjusting the mean of the worst-case mixture (with the largest variance) on the target value (Figure 5b, c). Again, the last step is a check whether the design can accommodate the worst-case variation without failure. This is a powerful method for delivering robust designs and processes. An important step of the proposed method is identifying the source of variation whose removal yields the largest decrease in the variance upper bound. In what follows, we present an algorithm for achieving this goal.

The algorithm incorporates two main steps, the first of which involves finding the source (or the pair of sources) sampling from which yields the variance upper bound. If sampling from a single source yields the largest variance, removing this source yields the largest decrease in the variation of the selected property. If the largest variance is obtained from sampling a pair of sources, for example sources $k$ and $s$, the largest decrease in the variation of the property is obtained by removing one of the two sources.

Indeed, if the variance upper bound is attained from sampling the pair of sources $(k,s)$, the removal of any other source $i$, $i \neq k, i \neq s$ outside the pair $(k,s)$ will not decrease the maximum variance, which will remain the same until the pair $(k,s)$ is destroyed by removing one of the two sources.
Which source will be removed \((k\) or \(s)\) depends on which source yields the largest variance if sampled with any of the remaining sources (outside the pair \(k,s\)) or sampled alone. Suppose that source \(s\), if sampled with source \(p\), produces the largest variance. Removing source \(s\) then yields the largest decrease in the variation of the property.

The algorithm for identifying the source whose removal yields the largest decrease in the variance upper bound can be illustrated by a numerical example. Suppose that five sources of a selected property are characterized by individual distributions of the property with variances \(V_1 = 208\), \(V_2 = 240\), \(V_3 = 108\), \(V_4 = 102\), \(V_5 = 90\) and means as \(\mu_1 = 39\), \(\mu_2 = 43\), \(\mu_3 = 45\), \(\mu_4 = 56\) and \(\mu_5 = 65\), respectively. The global maximum of the variance of the mixture distribution composed by these sources is \(V_{\text{max}} = 323\), attained from sampling the fifth source with probability \(p_5 = 0.41\) and the first source with probability \(p_1 = 0.59\). The second largest variance \(V_{\text{max}} = 297.62\) is attained from sampling the fifth source with probability \(p_5 = 0.345\) and the second source with probability \(p_2 = 0.655\). Removing the fifth source will reduce the variance upper bound below \(V_{\text{max}} = 297.62\). Calculations using the algorithm from [7] show that after removing the fifth source, the variance upper bound becomes \(V_{\text{max}} = 241.4\), obtained from sampling the fourth source with probability \(p_4 = 0.09\) and the second source with probability \(p_2 = 0.91\).

5. Applications of the new method for developing robust designs, assemblies and processes

*Increasing the robustness of operations*

Suppose that several operators (e.g., from different shifts) are setting the value of a critical parameter (e.g., length). The parameter for example, could be a critical distance that defines the position of a part in a manufacturing cell (e.g., the position of the glass in a glazing cell of a car manufacturing plant).

Each operator sets the critical distance with a particular mean and a standard error. In order to achieve the necessary consistency of production control, the production process must accommodate the worst variation of the set position from its target value. The worst variation of the set position can be determined by using the VUBT. After defining the pair of operators associated with the worst-case variation of the set position, the distance \(t_0\) between the mean of the worst-case mixture and the target value is determined (Fig.5 b). By altering the reference point associated with the start of the measurement (common to all operators), the set position can be easily shifted (increased or decreased) and this helps to set a process with the worst variation from the target value. The operation can now be made resistant against this worst-case variation.

*Increasing the robustness of mechanical and electronic components*

In the assembly from Fig.6a, component \(A\) with a mean diameter \(d\) must fit into component \(B\) with mean diameter \(D\). In order to guarantee precision, the clearance \(\Delta = D - d\) should not deviate significantly towards values greater than its optimum value \(\Delta_{\text{opt}}\) (Fig.6b). On the other hand, in order to avoid a fit failure (inability to fit \(A\) into \(B\)) or
jamming because of insufficient clearance, the difference $\Delta = D - d$ cannot deviate significantly towards values smaller than the optimum value $\Delta_{\text{opt}}$ (Fig.6b). The assemblies are manufactured on M machine centres (Fig.7). Each machine centre is associated with a particular precision during manufacturing the diameters $d$ and $D$. As a result, the reliability-critical clearance $\Delta = D - d$ varies from center to center and this is a source of unreliability. Failures are caused by unfavourable tolerance stacks due to excessive variation. Suppose that the ranges within which the diameters $d$ and $D$ vary are $d_{\text{min}} \leq d \leq d_{\text{max}}$ and $D_{\text{min}} \leq D \leq D_{\text{max}}$. Jamming for example occurs if, in an assembly, the outside diameter $d$ has deviated towards its upper limit $d_{\text{max}}$ and simultaneously, the inside diameter $D$ has deviated towards its lower limit $D_{\text{min}}$. Conversely, the assembly is imprecise if the outside diameter $d$ has deviated towards its lower limit $d_{\text{min}}$ and the inside diameter $D$ has deviated towards its upper limit $D_{\text{max}}$. Decreasing the variation of the clearance $\Delta = D - d$ will improve the reliability of the assembly. Usually, the variances of the gap $\Delta = D - d$ characterizing the manufacturing centers (Fig.7) producing the assemblies are known.

Then the pair of manufacturing centres yielding the maximum variance of the clearance $\Delta = D - d$ can be determined. If no source of variation (manufacturing centre) can be removed, the pair of distributions yielding the worst possible variation of the clearance can be identified. Next, the distance between the mean of the worst-case distribution and the optimal target value can be determined. The clearance from all manufacturing centres is then simultaneously increased or decreased by this value. This can for example be achieved by increasing/decreasing the inner diameter $D$ of component $B$ or decreasing/increasing the outer diameter $d$ of component $A$. A check is finally performed whether the design can accommodate the worst-case variation.

If sources of variation (manufacturing centres) can be removed, by removing the source resulting in the most significant decrease in the variance upper bound, the variability of the clearance $\Delta = D - d$ will be reduced and the capability index of the manufacturing process increased. This is a way of achieving a robust manufacturing
process, whose variation is under control, irrespective of the actual mixing proportions from the manufacturing centers.

\[ p_1 + p_2 + \ldots + p_M = 1 \]

Figure 7: The maximum variation of properties of an assembly manufactured from multiple machine centres is obtained from sampling not more than two machine centres.

Another example related to mechanical assemblies involves mechanical latches and switches where a mechanical spring delivered from different suppliers is a critical part of the assembly. The required force \( F_m \) exerted by the spring can be set by the manufacturer by selecting an appropriate initial deformation \( \Delta l_0 \). As a result, the variation of the spring force \( F = k \times \Delta l_0 \) depends only on the variation of the stiffness \( k \) of the spring from the individual suppliers. If the assembly is to operate properly, the stiffness of the spring should not vary excessively. A variation towards a higher stiffness will increase the spring force \( F \), while a variation towards a smaller stiffness will diminish it. It is important to select a set of suppliers yielding a small variation of the spring stiffness.

The next example is related to a mechanical application of the variance upper bound theorem in the case where the mean of the properties contributed from the separate suppliers is not critical. The example involves mechanical assemblies where the coefficient of thermal expansion \( \alpha \) of similar components from different suppliers must not vary significantly. Since the linear expansion \( \Delta l = \alpha l \Delta T \) is proportional to the coefficient of thermal expansion \( \alpha \), excessive variation in \( \alpha \), will induce excessive thermal stresses during a variation of temperature.

The last example features heat treatment centers delivering tempered steel parts after quenching. An important control parameter here is the strength after tempering, which correlates well with the hardness level. The hardness can be easily adjusted to a specified target value \( S_{opt} \) by varying the tempering temperature and duration. Large variations outside the target level \( S_{opt} \) lead to substandard items. Tempering the steel to insufficient strength \( S \leq S_{opt} - \Delta S_1 \), reduces its failure resistance. Tempering to a too high strength \( S > S_{opt} + \Delta S_2 \) increases the sensitivity of the steel to stress raisers (oxide inclusions, grooves, corners, steps, etc.) and again reduces the failure resistance. Furthermore, with increasing strength, fracture toughness \( K_c \) is reduced, the critical flaw size \( a_c \), decreases and may become smaller than the detection limit of the available non-destructive inspection technique. A flaw size below
the detection limit will be missed during inspection and will make the steel unsafe to use. This can be seen easily for a tensile opening mode of an edge crack with size \( a \). In this case, fast fracture occurs when the stress intensity factor \( Y\sigma\sqrt{\pi a} \) (the driving force behind the crack extension) becomes equal to or greater than the fracture toughness \( K_{lc} \) [13]

\[
Y\sigma\sqrt{\pi a} = K_{lc}
\]

where \( Y \) is the geometry factor, and \( \sigma \) is the loading stress. A reduction of the fracture toughness \( K_{lc} \) associated with increased strength means reduction of the critical flaw size

\[
a_c = \frac{1}{\pi \left[ K_{lc} / (Y\sigma) \right]^2}
\]

**Increasing the robustness of electronic devices**

Components building electronic circuits are characterized by properties like resistance, capacitance, inductance, etc. Because of imprecision during manufacturing, the actual magnitudes of these properties deviate from the stated nominal values.

Suppose that these components are part of safety-critical systems containing sensors measuring temperature, pressure, concentration, etc. in two different zones. A difference exceeding a particular threshold triggers an alarm or a shutdown system. Large deviations in the properties of the components building the circuit are undesirable, because they lead to a deteriorated performance of the safety-critical devices.

The components are manufactured by different centers/suppliers. Each center/supplier is characterized by its individual distribution of the corresponding property. Usually, the variation of the property (resistivity, capacitance, inductance, etc.) associated with the common pool of manufactured components for specified mixing proportions from the suppliers is not the maximum possible variation that can occur. There exists a particular combination of sources and mixing proportions that yields the largest (worst-case) variation. The VUBT makes it possible to calculate this worst-case variation and this will be illustrated by a simple example.

Suppose that electronic components are delivered from four suppliers. The mean resistances \( [\Omega] \) characterizing the individual suppliers are:

\[
\mu_R = \{500, 504, 510, 516\}
\]

The variances characterizing the individual suppliers are:

\[
\sigma_R = \{102, 141, 166, 85\}
\]

Suppose that the market shares of the suppliers are: \( p_R = \{0.15, 0.65, 0.15, 0.05\} \), where

\[
\sum_{i=1}^{4} p_{R_i} = 1.
\]

For the variance of the supplied components, equation (3) yields \( \nu = 150 \). A calculation of the maximum variance however by using the algorithm in [7] reveals \( \nu_{\text{max}} = 169 \), attained from sampling two suppliers only: the first supplier with a mixing proportion (market share) \( p = 0.18 \) and the third supplier with a mixing proportion (market share) \( q = 1 - p = 0.82 \). The designer must make sure that the electronic circuit will operate satisfactorily under the worst possible combination of suppliers yielding the maximum possible variation of the resistance.
Here, we need to point out that the distribution of properties from several suppliers is a distribution mixture, different from the individual distributions characterizing the separate suppliers or any particular distribution. For the example discussed earlier, the resultant mixture distribution is different from Gaussian distribution even if the resistance of the components from each individual supplier follows a normal distribution.

6. A virtual testing method for determining the probability of a faulty assembly

The design parameters related to any product are associated with uncertainty. This is usually caused by variability associated with the external loads acting on the product, the environment where the product operates, the technological processes used in the production of materials, the manufacturing processes and the operation. Variability of the input parameters characterizing a particular design transforms into variability of the output properties of the products. Because of the natural variation of design parameters, particular combinations of parameter values are transformed into undesirable deviations of the output properties from their optimal values which define a fault. Faults lead to deteriorated performance and failure.

A typical example of a fault caused by the variation of input design parameters can be given with interference fits (press or shrink fits) [14] that consist of a hub and a shaft (Fig.8).

![Figure 8: The press fit must be capable of carrying torque and axial force without slippage.](image)

This assembly must be capable of carrying torque and axial forces without slippage. All design parameters are associated with a physical variation after manufacturing but the two design parameters that affect most significantly the load-carrying capability of the assembly are the coefficient of friction $\mu$ between the hub and the shaft and the interference $\delta = 2\Delta r$ between the diameters of the hub and the shaft [5, 15]. These parameters are important because their variation affects most significantly the friction force $f$ per unit contact area. This is defined as $f = \mu p$ where $p$ is the contact pressure on the shaft (Fig.8). The variations of the coefficient of friction $\mu$ affect the friction force directly while the variations of the interference $\delta$ affect the friction force through the contact pressure $p$. If the temperature also varies, the interference is further affected, because of the different coefficient of thermal expansion of the hub and the shaft.

If, for example, the coefficient of thermal expansion of the hub is greater than the coefficient of thermal expansion of the shaft, with increasing temperature, the interference, the contact pressure and the load carrying capability of the assembly will decrease. Suppose that the friction coefficient varies in the interval $\mu_{\text{min}} \leq \mu \leq \mu_{\text{max}}$, the interference
varies in the interval $\delta_{\text{min}} \leq \delta \leq \delta_{\text{max}}$ and the temperature varies in the interval $T_{\text{min}} \leq T \leq T_{\text{max}}$. Suppose also that with increasing temperature, the coefficient of friction monotonically decreases. If for a particular interference (press fit) assembly, an elevated temperature is combined with a small coefficient of friction and a small value of the interference $\delta$ at a room temperature, the assembly could lose its capability to carry the specified torque and axial force.

In the cases where the joint distribution describing the variation of the design parameters is known, the link between the uncertainty in the design parameters from multiple sources and the probability of a faulty assembly due to unfavourable combinations of parameter values can be investigated by a Monte Carlo simulation. For a press fit working at the higher end $T_{\text{max}}$ of the temperature range, the probability of a faulty assembly can be found by a process referred to as virtual testing. This is essentially a simulation involving sampling values for the reliability-critical friction coefficient and interference, from their joint distribution at temperature $T_{\text{max}}$. After each sampling, a check is performed whether an assembly characterized by the sampled reliability-critical parameters can transmit the required loads without slippage. The ratio of the number of simulations resulting in a faulty assembly and the total number of simulations gives the likelihood of a faulty assembly.

The variation of many design parameters may not affect significantly the reliability of a product. The virtual testing begins with identifying the variation of which design parameters affects most severely the reliability of the component/assembly. These are reliability-critical parameters. The algorithm of the virtual testing method can be generalized for an arbitrary number of reliability-critical parameters and here is its description in pseudo-code.

Algorithm 1

```plaintext
Algorithm 1

x[n]:
/* Global array containing the current values of the n reliability-critical design parameters */
Fault_counter = 0;
For i = 1 to Number_of_trials do
{ /* Generate the i-th set of n reliability-critical parameters by sampling their joint
distribution and placing them in the array x[] */
    Sample_all_reliability_critical_design_parameters();
    /* Check whether the assembly is faulty by using the current values of the reliability-critical
    parameters in the array x[n]. For a particular combination x[1],...,x[n] of values, the function returns 1 or 0 depending on whether
    a fault is present or not */
    Fault = Is_faulty_assembly();
    /* If a faulty assembly is present, then increment the failure counter */
    If (Fault==1) then  Fault_counter = Fault_counter+1;
}
Probability_of_faulty_assembly = Fault_counter / Number_of_trials;
```
In the simulation loop controlled by the variable \(i\), the procedure \texttt{Sample\_all\_reliability\_critical\_design\_parameters()} is called, whose purpose is to generate realizations for all reliability-critical design parameters by sampling their joint distribution. If the reliability-critical parameters are statistically independent, realizations are generated by a sequential sampling of their individual distributions. After obtaining a set of values for the parameters controlling the reliability of the assembly, the function \texttt{Is\_faulty\_assembly()} is called to perform design calculations and check whether the set of values for the reliability-critical parameters defines a faulty assembly. If this is so, the fault counter is incremented. At the end of the simulation trials, the probability of a faulty assembly is obtained as a ratio of the number of simulated faulty assemblies to the total number of simulated assemblies.

The application of this approach will be illustrated by assessing the probability that an assembly of the type in Figure 6 will exhibit a fault during operation at unknown temperature conditions. The reliability-critical design parameters characterizing the assembly are the diameters \(d\) and \(D\) of components \(A\) and \(B\) at room temperature and the working temperature \(t\). The reason is that these three parameters fully determine the magnitude of the diameter clearance \(\Delta\), whose excessive deviation from the optimum value \(\Delta_{opt}\) causes failure.

Suppose that the diameters \(d_0\) and \(D_0\) at temperature \(t_0 = 20\) °C follow normal distributions with means \(\mu_{d0} = 55\text{mm}\), \(\mu_{D0} = 55.5\text{mm}\) and standard deviations \(\sigma_{d0} = 0.12\text{mm}\), \(\sigma_{D0} = 0.22\text{mm}\). The optimal diameter clearance of the assembly is \(\Delta_{opt} = 0.5\text{mm}\). Suppose that if the diameter clearance falls below \(\Delta_{min} = 0.1\text{mm}\) jamming occurs and if the clearance exceeds \(\Delta_{max} = 0.8\text{mm}\), precision of operation is lost. The coefficients of thermal linear expansion of the materials are \(\alpha_A = 24 \times 10^{-6} \text{K}^{-1}\) for component \(A\) and \(\alpha_B = 11 \times 10^{-6} \text{K}^{-1}\) for component \(B\). The working temperature of the assembly can be anywhere in the temperature interval \(-55\) °C \(\leq t \leq 400\) °C. Consequently, the temperature \(t\) is assumed to be uniformly distributed in this temperature interval.

Since all reliability-critical design parameters are statistically independent, sampling from their joint distribution is equivalent to a sequential sampling from their individual distributions. The outlined algorithm transforms into the following algorithm:

\begin{verbatim}
Fault_counter = 0;
For i = 1 to Number_of_trials do
{
    d0 = Sample_diameter_comp_A();
    D0 = Sample_diameter_comp_B();
    t = Sample_temperature();
    \(\Delta t = t - t_0\);
    d = d0(1 + \alpha_A \Delta t)
    D = D0(1 + \alpha_B \Delta t)
    if \((D - d < \Delta_{min} \text{ or } D - d > \Delta_{max})\) then
       Fault_counter = Fault_counter + 1;
}
\end{verbatim}
Fault_counter = Fault_counter+1;
}
Probability_of_faulty_assembly = Fault_counter / Number_of_trials;

Initially, instances $d_0$ and $D_0$ of the diameters of components $A$ and $B$ at $t_0 = 20 \degree C$ are calculated, by sampling from their individual distributions. Next, a uniformly distributed temperature in the interval $(t_{\min}, t_{\max})$ is generated, by using the linear transformation $t = t_{\min} + (t_{\max} - t_{\min}) \times u$, where $u$ is a random number, uniformly distributed in the interval $(0,1)$. The temperature change is then determined from $\Delta t = t - t_0$. After determining the thermal expansions $d = d_0(1 + \alpha_A \Delta t)$ and $D = D_0(1 + \alpha_B \Delta t)$ of the diameters at temperature $t$, a check is performed whether $D - d < 0.1$ or $D - d > 0.8$. If any of these is fulfilled, the assembly will fail and the fault counter is incremented. The probability of a faulty assembly for the given set of input data has been calculated by implementing the outlined algorithm in C/C++. The empirical probability of a faulty assembly has been determined to be 11%, obtained on the basis of 100000 simulation trials.

This approach can be extended in the case of assemblies coming from multiple sources. In what follows, a Monte Carlo simulation algorithm in pseudo-code is presented for revealing the probability of a faulty assembly from multiple sources. The worst variation of the output property characterizing the assemblies is determined by using the VUBT.

**Algorithm 2**

```plaintext
i=0;
Repeat
Select a source with index i;
/* $x_i[n_i]$ is a global array containing the current values of $n_i$ reliability-critical random parameters characterizing the assembly coming from the i-th source */
/* $y_i[n_i]$ is a global dynamic array containing the current values of the output property characterizing the assembly coming from the i-th source */
For $m = 1$ to Number_of_trials do
{
/* Generates the m-th current set of reliability-critical parameters; Sampling from their joint distribution produces a value (realization) for each parameter. The sampled values are stored in x_i[] */
Sample_Reliability_critical_parameters();
/* Performs design calculations and calculates a value of the output property for the current set of sampled values for the reliability-critical parameters x_i[j]; the result is stored in the y_i[]-th dynamic array */
y_i[m] = Calculate_output_property();
}
Sort the values of the y_i[]-th array in ascending order;
Until (i == M);  // Until all sources have been scanned
/* Finds the pair of distributions y_s[], y_k[] and the mixing proportion $p$, characterized by the largest variance by using the procedure described in [7] */
```
Find_worst_property_variation();

Fault_counter = 0;
For i = 1 to Number_of_trials do
{
    /* Generates a random number, uniformly distributed between 0 and 1 */
    u = Real_random();
    If (u < p) then Output = Sample_distribution \( y_k \);
    else Output = Sample distribution \( y_s \);

    /* Checks if the sampled output property defines a faulty assembly */
    Fault = Is_faulty_assembly(Output);
    If (Fault == 1) then Fault_counter = Fault_counter+1;
}
Probability_of_faulty_assembly = Fault_counter / Number_of_trials;

The first part of the algorithm creates the distribution of the output property for the \( i \)-th individual source, based on the variation of the reliability-critical parameters characterizing the source. For each simulation trial, the procedure Sample_reliability_critical_parameters(); assigns random values to the reliability-critical parameters by sampling their joint distribution. If the reliability-critical parameters are statistically independent, the realizations are created by a sequential sampling from the individual distribution of each parameter. Next, design calculations are performed with the procedure Calculate_output_property() and the current result for the output property is stored in the dynamic array \( y_i \). In this way, the variation of the output property is determined for all sources. Each of the dynamic arrays is sorted in ascending order to provide an easy sampling of the cumulative distribution characterizing the output property.

The second part of the algorithm starts with the procedure Find_worst_property_variation() that determines the pair of sources yielding the worst variation of the output property. The software implementation of this procedure can be found in [7].

In the simulation loop controlled by the variable \( i \), a sampling from the worst-case variation of the output property is conducted. This is effectively a sampling from a mixture distribution containing at most two individual distributions. Suppose that sampling from the distribution mixture

\[
F(x) = pF_k(x) + qF_s(x)
\]  \( (11) \)

is required, where \( p \) and \( q = 1 - p \) are the mixing proportions yielding the largest variance and \( F_k(x) \), \( F_s(x) \) are the individual distributions composing the mixture. Sampling the distribution mixture (11) involves two steps: (i) a random selection of an individual distributions to be sampled and (ii) a random sampling from the selected individual distribution. In the described algorithm, a random selection with probability \( p \) of individual distribution \( F_k(x) \) and with probability \( 1 - p \) of the individual distribution \( F_s(x) \) has been done by simulating a uniformly distributed random variable \( u \) in the interval \((0,1)\) by the standard procedure Real_random(), whose software implementation can be found in
most books on Monte Carlo simulation. After generating the random variable \( u \), a check is performed. If \( u \leq p \) is fulfilled, the distribution \( F_1(x) \) is selected, if \( u > p \), the distribution \( F_3(x) \) is selected. After selecting one of the distributions, a random sampling from the selected distribution is performed by using a standard method.

After obtaining a sample from the worst-case variation, the function `is_faulty_assembly()` is called to check whether the sampled output parameter defines a faulty assembly. If a faulty assembly is present, the fault counter is incremented. Again, at the end of the Monte Carlo trials, the probability of a faulty assembly is obtained as a ratio of the value stored in the fault counter and the total number of simulation trials.

This approach will be illustrated by a simple example. Suppose that the electrical resistance of the components from the individual suppliers of electronic components from the example in the previous section follows a normal distribution with means and variances as in the example. The mixture distribution \( F(x) \) yielding the largest variance is obtained from sampling the first and the third supplier:

\[
F(x) = pF_1(x) + qF_3(x)
\]

where, \( p = 0.18 \), \( q = 1 - p \), \( F_1(x) \) and \( F_3(x) \) are the individual Gaussian distributions characterizing the first and the third supplier.

Here, we point out again that the distribution \( F(x) \) is a not a normal distribution. It is a distribution mixture with mean

\[
\mu = p\mu_1 + q\mu_3 = 0.18 \times 500 + 0.82 \times 510 = 508.2
\]

where, \( \mu_1 = 500 \), \( \mu_3 = 510 \), and variance

\[
\sigma^2 = p\sigma_1^2 + q\sigma_3^2 + pq(\mu_1 - \mu_3)^2 = 169.2
\]

where \( \sigma_1^2 = 102 \), \( \sigma_3^2 = 166 \).

Suppose now that two electronic components are assembled in two symmetrical sections of an electronic device and a difference in the resistance of more than 40\( \Omega \) essentially means ‘a faulty device’. The percentage of faulty devices has been calculated by using the outlined algorithm implemented in C/C++. The empirical probability of a faulty device was 2.9\%, obtained on the basis of 100000 simulation trials.

7. Conclusions

The following conclusions can be drawn:

- A new method for increasing the robustness of products, processes and operations has been presented. The robustness can be improved by a design based on the worst-case variation of the design parameters, determined by using the variance upper bound theorem. In this respect, a number of applications related to increasing the robustness of mechanical and electronic components and devices based on the variance upper bound theorem have been discussed.
- For sources of variation that can be removed, an efficient algorithm has been proposed for determining the source whose removal yields the largest decrease in the worst-case variation of properties.
- A new, non-parametric estimate has been proposed for the capability index of a process whose output is affected by multiple sources of variation. The non-parametric estimate is of significant importance to statistical process control.
An algorithm has been proposed for determining the likelihood of a faulty assembly by virtual testing based on sampling from the variation of the reliability-critical design parameters.

- A new method for virtual testing of assemblies has been proposed, based on the variance upper bound theorem. The method determines the probability of a faulty assembly from multiple sources.

References

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Todinov, Michael has his background in Mechanical Engineering and Engineering Mathematics and his research area is Reliability and Risk. Before joining Oxford Brookes University in 2007 as a professor in Mechanical Engineering, he was head of risk and reliability in the School of Applied Sciences at Cranfield University. He has authored two books and numerous research papers in the area of reliability and risk. A significant part of his research has been funded by the nuclear, aerospace, automotive and offshore oil and gas industry. He holds a PhD in the area of mathematical modelling of temporal and residual stresses and a higher doctorate, DEng, in the area of reliability and risk modelling. His name is associated with results in the area of risk-based reliability analysis, reliability analysis and optimization of complex systems, stochastic flow networks, statistics of inhomogeneous media, statistical theory of failure locally initiated by flaws and models for rational risk decisions.