Additive Weibull Model for Reliability Analysis

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Abstract: Statistical distributions of time to failure for many mechanical systems reveal bathtub-shaped failure rate in practice. Bathtub shaped failure rate is very much useful in reliability engineering for the determination of burn-in and plays key role in provisions of warranty. The traditional Weibull distribution can cover the profile piecewise only and there are very few practical models to model bathtub-shaped failure rate. The Additive Weibull model discussed in this paper is based on adding two Weibull survival functions. Some simplifications of the model are presented and the parameter estimation methods based on the graphical estimation technique are discussed. Various case studies discussed in this paper illustrate the applicability of the model. Results and discussions on cases gave some interesting utilities, which are also presented in this paper.

Keywords: Traditional Weibull distribution, bathtub profile, Weibull probability plot, parameter estimation, burn-in

1. Introduction

The Weibull distribution is one of the most widely used probability distributions in the reliability engineering discipline. It was developed in 1937 by Dr. Waloddi Weibull. It was introduced to a greater population in 1951 through his paper “A Statistical Distribution Function of Wide Applicability”. It is flexible in modeling failure time data as the corresponding failure rate function can be increasing, constant or decreasing. If a system’s hazard rate assumes a bathtub profile, rather it becomes easier for many decision-making issues related to reliability and maintenance engineering. As traditional Weibull covers the bathtub profile piecewise, obtaining a mathematical model delineating the bathtub profile will be of great help. Few models developed are reported in literature, selected few of which is presented in the following section.

2. Literature Review

Hjorth [1] proposed a three-parameter distribution with increasing, decreasing, constant or bathtub-shaped failure rate function. This paper mainly attempts description of hazard rate function piecewise. Mudholkar and Srivastava [2] proposed an exponentiated Weibull distribution which reveals the consolidated view of the failure rate phenomenon. Authors used lifetime data of 50 devices as a case.
Chen [3] proposed two parameter lifetime distributions to capture bathtub shaped hazard rate and used Type II censored data for validation. Wang [4] proposed a model based on adding two simple Burr XII distributions. Xie [5] proposed a new model referring to as modified Weibull extension. This is a three-parameter model, which exhibits the bathtub profile in entirety. Xie and Lai [6] presented additive model involving four parameters to describe the bathtub profile. The parameters of this model were estimated using graphical method. Jha [7] discussed the problem of determining optimal burn-in time under general failure modes. The author provided bounds for optimal burn-in considering bathtub shaped failure rate function as a special case.

3. Problem on Hand

Traditional Weibull distribution of course describes bathtub shaped hazard rate function, but piecewise. But many of managerial decisions on production facilities can effectively be taken if a bathtub profile hazard rate function is fitted to the time to failure data of the machine under investigation. But Jiang and Murthy [8] observed that often when the failure data are plotted on WPP, the points are scattered along a smooth curve rather on a straight-line whereby it loses accuracy. This is because many failure modes are mixed up and such a thing is known as “competing failure mode”. Therefore, a mechanism to screen and yield appropriate parameter is utmost required. Getting effective burn-in is the main aim in warranty modeling. Actually mixed data will rather fit a bathtub shaped hazard rate by isolation or screening. Thus by getting a model fitting a bathtub shaped hazard rate function will help resolving these issues. This model can be referred to as competing failure rate model. This paper discusses an additive Weibull distribution which results in bathtub shaped hazard rate function and applied to various cases. The findings of the cases are discussed in this paper.

4. Additive Weibull

The additive Weibull model proposed by Xie combines two traditional Weibull distributions; one with an increasing hazard rate and another with decreasing hazard rate. The cumulative hazard function is as expressed in Equation (1). Respective reliability and hazard rate functions are as given in Equations (2) and (3).

\[ H(t) = \int_0^t h(t)dt = (at)^b + (ct)^d, \quad t, a, c \geq 0, \quad b > 1 \text{ and } d < 1 \]  

(1)

\[ R(t) = e^{-\int_0^t h(t)dt} = e^{-H(t)} = e^{[-(at)^b + (ct)^d]} \]  

(2)

\[ h(t) = ab(at)^{b-1} + cd(ct)^{d-1} \]  

(3)

4.1 Analytical Investigation

We know that, \( h(0) = \infty \) and \( h(\infty) = \infty \). Hence, it poses the possibility of at least one minimum. Further proceeding with conditions of monocity,

\[ h'(t) = t^{d-2}[a^2 b(b-1)t^{b-1} - c^2 d(1-d)] \]  

(4)

Equating Equation (4) to zero gives the stationary value, which is
Additive Weibull Model for Reliability Analysis

\[ t = t_0 = \left( \frac{d \cdot (1 - d)}{b \cdot (b - 1)} \right)^{\frac{1}{b - d}} \]  \hspace{1cm} (5)

It is clear from Equation (4) that \( h'(t) \) is negative when \( t < t_0 \) and it becomes zero at \( t = t_0 \) and \( t > t_0 \) it takes positive value. Moreover since \( h(0) = \infty \) and only one stationary value, \( h(t) \) should decrease from in the rage of \( t \in [0, t_0] \). Similarly as \( h(\infty) = \infty \), \( h(t) \) should increase from \( h(t_0) \) to \( \infty \). \( h(t_0) \) can be obtained after simplification using Equations (3) and (5) can be expressed as

\[ h(t_0) = c^d \cdot d^{d-1} \cdot \frac{(b - d)}{(b - 1)} \]  \hspace{1cm} (6)

In a nutshell, \( h(t) \) has a minimum value at \( t = t_0 \). It means that it takes bathtub profile. Another interpretation would be that when \( t \) assumes small values, the second term of Equation (3) of the hazard rate function will be more dominant and describe the behaviour of hazard rate, which is a decreasing function. Similarly for larger \( t \), the first term becomes dominant, which is an increasing function. But for the medium values of \( t \) both the terms of Equation (3) become equally dominant will give rise to the flat middle portion as second term in Equation (3) has negative index. Thus the entire bathtub profile is represented in entirety.

5. Estimation Parameters of the Additive Weibull Model

Parameter estimation is usually a difficult problem for Weibull models as the functions are transcendental. Graphical approaches have been used extensively in determining whether a particular distribution is suited to modeling a given set of failure data. Plotting papers for several distributions are available or can be designed to carry this out. If plotted on WPP, fall roughly along a straight line, then the distribution selected is adjudged appropriate for modeling the data. Jiang observed that often when the failure data are plotted on WPP, the points are scattered along a smooth curve rather on a straight-line. This issue can be resolved using this additive model by segregating the patterns. Required probability plot can be developed as described below.

Taking log on Equation (2),

\[ \ln \left[ \frac{1}{R(t)} \right] = (at)^b + (ct)^d \]  \hspace{1cm} (7)

As mentioned earlier, for the case of small value of \( t \), the above equation can be approximated to

\[ \ln \left[ \frac{1}{R(t)} \right] \equiv (ct)^d \]  \hspace{1cm} (8)

\[ \ln \left[ \frac{1}{R(t)} \right] \equiv (at)^b \]  \hspace{1cm} (9)

Taking log again on Equations (8) and (9),

\[ \ln \ln \left[ \frac{1}{1 - F(t)} \right] \equiv d \cdot \ln(t) + d \cdot \ln(c), \text{ for small } t \]  \hspace{1cm} (10)

\[ \ln \ln \left[ \frac{1}{1 - F(t)} \right] \equiv b \cdot \ln(t) + b \cdot \ln(a), \text{ for large } t \]  \hspace{1cm} (11)

On log scale and \( F(t) \) on log log scale provides the probability plotting paper for this model. On this plot Equations (10) and (11) are straight lines with \( d \) and \( b \) as respective slopes, and \( d \ln(c) \) and \( b \ln(a) \) as intercepts respectively.
6. **Applicability**

A computer code is written in MATLAB, Version 7.0 to transform the given data to obtain the required probability plot. From the plot if it exhibits two distinct patterns, it can be segregated accordingly. Using the traditional Weibull probability plot, parameters can be estimated. Using the computer code the bathtub shaped hazard rate function can be obtained. The programme uses polyfit and polyval facilities to fit and extract parameters. This is illustrated using three real life cases in the following section.

6.1 **Case Studies**

This model has been applied on three real life cases. Normally data in the field is collected in two different ways. One is collecting the time to failure, which will be ranked. This is classified as small data. For a larger data, the time to failure collected will be transformed into frequency table. Cases considered here has the combination of both.

6.1.1 **Case 1**

The data set considered here is an actual set of failure time data collected during unit testing. The transformations required to obtain the probability plot are carried out as shown in Table 1. The probability plot obtained is as shown in Figure 1. Probability plots for segregated portions of the collected data are shown in Figures 2 and 3. Finally the parameters obtained are: $a = 0.1168$, $b = 2.5315$, $c = 0.1421$ and $d = 0.8876$. The bathtub shaped hazard rate function plot is shown in Figure 4.

6.1.2 **Case 2**

The data collected is as given in Table 2. Applying similar approach after due transformation on data, pattern of the data, Weibull probability plot for segregated portions and the bathtub shaped hazard rate function’s plot is shown in Figure 5. The parameters estimated are: $a = 0.0046$, $b = 1.8878$, $c = 0.0045$ and $d = 0.7306$

6.1.3 **Case 3**

Similarly for the Case 3, analyses are carried out on the actual data set of failure time of 280 washing machines of a leading company, situated in state of Goa, India. Carrying out the similar exercise the parameters obtained are: $a = 0.0679$, $b = 3.0345$, $c = 0.0652$, $d = 0.9508$ and relevant plots are as shown in Figure 6, for the data shown in Table 2.

7. **Results and Discussions**

The given data set collected will undergo probability plotting on the additive Weibull probability plot as discussed in analytical investigation. If this plot reveals mixed up pattern, then only further analysis will be carried out. In case of mixed pattern, the data is to be segregated into two parts, each of which will be tested on the traditional Weibull probability plot. Fitting a regression line the required parameters can be obtained. For the parameters obtained, the hazard rate function can be plotted which will give the bathtub shaped profile only when the straight line is ascertained in the subsequent plots. On the obtained bathtub shape, the change over point can be treated as effective burn-in.
Table 1: Failure data of 18 electronics devices

<table>
<thead>
<tr>
<th>Time to failure t</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freq</td>
<td>53</td>
<td>29</td>
<td>29</td>
<td>36</td>
<td>13</td>
<td>25</td>
<td>22</td>
<td>16</td>
<td>18</td>
</tr>
<tr>
<td>ln(t) (X-axis)</td>
<td>0.6931</td>
<td>1.0986</td>
<td>1.3863</td>
<td>1.6094</td>
<td>1.7918</td>
<td>1.9459</td>
<td>2.0794</td>
<td>2.1972</td>
<td></td>
</tr>
<tr>
<td>cdf, F(t)</td>
<td>0.17</td>
<td>0.263</td>
<td>0.356</td>
<td>0.472</td>
<td>0.514</td>
<td>0.594</td>
<td>0.665</td>
<td>0.717</td>
<td>0.774</td>
</tr>
<tr>
<td>lnln[1/(1-F(t))] (Y-axis)</td>
<td>-1.67</td>
<td>-1.1839</td>
<td>-0.8176</td>
<td>-0.4464</td>
<td>-0.3250</td>
<td>-0.1015</td>
<td>0.0911</td>
<td>0.2331</td>
<td>0.3996</td>
</tr>
<tr>
<td>Time to failure t</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
</tr>
<tr>
<td>Freq</td>
<td>8</td>
<td>22</td>
<td>11</td>
<td>13</td>
<td>5</td>
<td>5</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>ln(t) (X-axis)</td>
<td>2.30</td>
<td>2.3979</td>
<td>2.4849</td>
<td>2.5649</td>
<td>2.6391</td>
<td>2.7081</td>
<td>2.7726</td>
<td>2.8332</td>
<td>2.8904</td>
</tr>
<tr>
<td>cdf, F(t)</td>
<td>0.80</td>
<td>0.8714</td>
<td>0.9068</td>
<td>0.9486</td>
<td>0.9646</td>
<td>0.9807</td>
<td>0.9936</td>
<td>0.9968</td>
<td>0.9999</td>
</tr>
<tr>
<td>lnln[1/(1-F(t))] (Y-axis)</td>
<td>0.4</td>
<td>0.718</td>
<td>0.863</td>
<td>1.087</td>
<td>1.206</td>
<td>1.373</td>
<td>1.618</td>
<td>1.747</td>
<td>2.220</td>
</tr>
</tbody>
</table>

Fig. 1: Weibull probability plot of Case 1

The procedure developed was applied on three different cases; the summary results are depicted in Table 4. This way this model will serve the analyst in two ways:

i) Helps in getting effective burn-in period which can be declared as warranty period. Thus this model in a way works as warranty model

ii) From the bathtub shaped hazard rate function, the ageing region can be effectively used to draw preventive maintenance (PM) policy. If a relatively a flat region is exhibited, which can be used for ascertaining the useful life of the system. In this flat region, PM is not recommended. Thus this model takes care of mixed region.
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Fig. 2: Weibull probability plot for first six points of Case 1

Fig. 3: Weibull probability plot for last six points of Case 1

Fig. 4: Hazard rate plot of Case 1
8. Conclusions

Models with bathtub shaped hazard rate function are useful in reliability analysis and particularly in reliability related decision making and cost analysis. To this effect the additive Weibull model helps in capturing the bathtub shaped hazard rate function. First, the complete
data will be transformed into the probability plot. If this plot reveals any mixed pattern, the treatment will be continued. Next segregation based on the mixed pattern, each of which further put under the probability plot to estimate the additive Weibull parameters, using which the bathtub profile can be obtained. Using the profile obtained warranty can be modeled and maintenance decisions can be taken. Three different cases discussed illustrate the applicability of this model.

### Table 4: Results - summary of cases

<table>
<thead>
<tr>
<th>Case</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>h(t) plot</th>
<th>Effective burn-in</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1168</td>
<td>2.5315</td>
<td>0.1421</td>
<td>0.8876</td>
<td>Bathtub</td>
<td>0.96</td>
</tr>
<tr>
<td>2</td>
<td>0.0046</td>
<td>1.8878</td>
<td>0.0045</td>
<td>0.7306</td>
<td>Bathtub</td>
<td>31</td>
</tr>
<tr>
<td>3</td>
<td>0.0679</td>
<td>3.0345</td>
<td>0.0652</td>
<td>0.9508</td>
<td>Bathtub</td>
<td>1.19</td>
</tr>
</tbody>
</table>

References


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