Optimal Distribution of Constrained Resources in Bi-contest Detection-Impact Game

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Abstract: The paper considers a game between single attacker and single defender. In this game the attacker distributes his constrained resource between target detection and target destruction (impact) efforts, whereas the defender distributes his constrained resource between target counter-detection (camouflage, concealment and decoys) and counter-destruction (protection) efforts. In order to destroy the target the attacker should succeed in both target detection and impact contests. The success probability in each contest depends on the efforts of the agents and is determined by an attacker-defender contest success function. The attacker seeks to achieve the greatest target vulnerability (probability of destruction). The defender seeks to minimize the vulnerability. The paper studies the optimal resource distribution as solution of non-cooperative minmax game between the two agents.

Keywords: Game theory; Vulnerability; Defense; Attack; Protection; Detection; Contest success function

1. Introduction

When considering the risk of intentional attacks, it is important to realize that the use of an adaptive strategy allows attacker to act with maximal efficiency. Choosing the time, place, and means of attacks gives the attacker an advantage over the defender. Therefore, the optimal policy for allocating resources among possible defensive investments should take into account the attacker's strategy (Bier 2004, Azaiez and Bie 2007, Hausken 2002, Zhuang and Bier 2007, Hausken 2006, Levitin 2007).

Defending systems from intentional impacts usually presumes applying measures preventing both target detection and target destruction by the enemy. The first category of measures includes camouflage, concealment and decoys (CCD) that use various materials and techniques to hide, blend, disguise, decoy, or disrupt the appearance of targets. The second category includes various types of protections (from casings and bunkers to active anti-missile systems) aimed at reducing probability of target destruction in the case when it is detected and attacked.

Greater the CCD and the protection efforts, less are the target detection and destruction probabilities, respectively. However, usually the defense resources are limited and the defender must distribute them between the CCD and protection.
The intelligent attacker also has to distribute his limited resources between efforts aimed at detection of the target and its destruction. This distribution is aimed at achieving the greatest probability of target destruction. This paper suggests a model that assumes that the defender and the attacker participate in non-cooperative bi-contest game. In order to destroy the target the attacker must succeed in both target detection and impact contests.

The defender builds the system over time. The attacker takes it as given when he chooses his attack strategy. Therefore, we analyze a two period game where the defender moves in the first period, and the attacker moves in the second period. In this paper we consider the most conservative min-max defender's strategy that minimizes the maximal possible system vulnerability assuming that the attacker chooses the most harmful strategy in response to any defender's strategy. This is not only possible approach, but it is considered “particularly appropriate in the design of robust military systems” (Shier 1991).

2. The bi-contest model

2.1 Nomenclature

- $R, r$: total attacker's and defender's resources
- $X, x$: relative fraction of attacker's and defender's resources allocated to target detection contest
- $A, a$: costs of attacker's and defender's effort unit in impact contest
- $B, b$: costs of attacker's and defender's effort unit in target detection contest
- $T, t$: attacker's and defender's efforts in impact contest
- $S, s$: attacker's and defender's efforts in target detection contest
- $m$: attacker-defender impact contest intensity
- $f$: attacker-defender target detection contest intensity
- $h$: detection effort ratio parameter
- $g$: impact effort ratio parameter
- $w$: target detection probability (function of $x, X, h$ and $f$)
- $p$: conditional probability of target destruction given it is attacked (function of $x, X, g$ and $m$)
- $V$: target vulnerability

The total attacker's resource is $R$. The attacker can allocate part of his resource $RX (0 \leq X \leq 1)$ into the target detection effort and the remaining part of the resource $R(1-X)$ into the impact effort. The cost of the impact effort unit is $A$. The cost of the detection effort unit is $B$. Therefore the total impact effort the attacker makes is $T=R(1-X)/A$, whereas the total detection effort is $S=RX/B$.

The total defender's resource is $r$. This resource is distributed between preventing the attackers' detecting the target (CCD actions) $xr (0 \leq x \leq 1)$ and protecting the target against the attacker's impact $r(1-x)$. The cost of the protection effort unit is $a$. The cost of the CCD effort unit is $b$. The effort for protecting the defended object is $t = r(1-x)/a$, whereas the CCD effort is $s=rx/b$.

We assume that two contests take place in the considered game: target detection contest and impact contest. We here apply the commonly used ratio form of the attacker-
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defender contest function (Hausken 2005, Tullock 1980, Skaperdas 1996). The probability of the attacker's success in the target detection contest is

\[ w = \frac{S^f}{S^f + \frac{S}{X}} = \frac{(RX/B)^f}{(RX/B)^f + (RX/b)^f} = \frac{X^f}{X^f + h^f X^f} \]  

(1)

where \( f \) is a parameter that specifies the intensity of the contest, that is how decisively the agents fight or compete in the contest. \( f=0 \) gives egalitarian distribution (the magnitudes of the agents' efforts do not affect the outcome of the contest). \( f=1 \) gives proportional distribution. \( f=\infty \) gives winner-take-all outcome (even smallest superiority of one of agents guarantees his success).

It should be noted that in the case of equivalent efforts \( S=s \) no one of agents achieves superiority and the outcome of the contest is determined by chance: \( w=0.5 \). The equality of efforts does not mean the equality of allocated resources. To achieve the effort equality the agents usually have to allocate considerably different resources. This is expressed by the detection effort ratio parameter \( h = \frac{B}{b R} \) that specifies how the relative resource ratio \( x/X \) is realized into effort ratio \( s/S \) in the target detection contest:

\[ \frac{s}{S} = h \frac{x}{X} \]  

(2)

\( h>1 \) gives advantage to the defender, whereas \( h<1 \) gives advantage to the attacker.

The probability of the attacker's success in the impact contest (given the target is detected) is

\[ p = \frac{T^m}{T^n + t^{m}} = \frac{(1-X)^n}{(1-X)^n + [r(1-x)/a]^n} = \frac{(1-X)^n}{(1-X)^n + g^n (1-x)^n} \]  

(3)

where \( g = \frac{A}{a R} \) is the impact effort ratio parameter that specifies how the relative impact resource ratio \((1-x)/(1-X)\) is realized into impact effort ratio \( t/T \) in the impact contest:

\[ \frac{t}{T} = g \frac{1-x}{1-X} \]  

(4)

\( g>1 \) gives advantage to the defender, whereas \( g<1 \) gives advantage to the attacker.

In order to destroy the target the attacker has to succeed in both target detection and impact contests. Therefore, the target vulnerability (probability of the target destruction) is equal to the product of probabilities of the attacker's success in the both contests:

\[ V(x,X) = wp = \frac{X^f (1-X)^m}{(X^f + h^f X^f)[(1-X)^m + g^m (1-x)^m]} = \frac{1}{1 + h^f \left( \frac{x}{X} \right)^f + g^m \left( \frac{1-x}{1-X} \right)^m} \]  

(5)

The attacker tries to maximize the vulnerability by choosing proper resource distribution between the target detection effort and the impact effort. This distribution is determined by variable \( X \). The defender should choose his resource distribution \( x \) that minimizes the target vulnerability. This constitutes a non-cooperative minmax game between the attacker and the defender. We consider a two period game where the defender moves first and the attacker second. The defender's free choice variable is his resource distribution \( x \) chosen in the first period, and the attacker's free choice variable is his resource distribution \( X \) chosen in the second period. The attacker takes the defender's
obtain.

Dividing both sides of the first equation by the corresponding sides of the second one we get

to prove rigorously that the obtained point is the saddle point of function $V(x,X)$, that can be obtained solving the system of equations (Straffin 1993)

\[
\begin{align*}
\frac{\partial V}{\partial x} &= 0, \\
\frac{\partial V}{\partial X} &= 0.
\end{align*}
\]  

(to prove rigorously that the obtained point is the saddle point of function $V(x,X)$ one has to prove that the Hessian matrix of $V(x,X)$ is indefinite. Whereas the analytical proof is too complicated, the analytical results obtained below was confirmed by a numerical optimization procedure).

The corresponding derivatives take the form

\[
\frac{\partial V}{\partial x} = \frac{h'(x)g'(x) 
\left[ 1 + g^m \left( \frac{1 - x}{1 - X} \right)^m \right] 
- h(x) \left[ 1 + h^m \left( \frac{1 - x}{1 - X} \right)^m \right]}{1 + h^m \left( \frac{1 - x}{1 - X} \right)^m} \]  

\]  

\[
\frac{\partial V}{\partial X} = \frac{g^m \left( \frac{1 - x}{1 - X} \right)^m \left[ 1 + h^m \left( \frac{1 - x}{1 - X} \right)^m \right]}{1 + h^m \left( \frac{1 - x}{1 - X} \right)^m} \]  

\]  

System (6) can be rewritten now in the form

\[
\begin{align*}
\left[ g^m \left( \frac{1 - x}{1 - X} \right)^m \right] \left[ 1 + h^m \left( \frac{1 - x}{1 - X} \right)^m \right] 
- h(x) \left[ \frac{1 + h^m \left( \frac{1 - x}{1 - X} \right)^m}{1 + h^m \left( \frac{1 - x}{1 - X} \right)^m} \right] 
\end{align*}
\]  

\]  

\[
\begin{align*}
\left[ g^m \left( \frac{1 - x}{1 - X} \right)^m \right] \left[ 1 + h^m \left( \frac{1 - x}{1 - X} \right)^m \right] 
= h(x) \left[ \frac{1 + h^m \left( \frac{1 - x}{1 - X} \right)^m}{1 + h^m \left( \frac{1 - x}{1 - X} \right)^m} \right]
\end{align*}
\]  

Dividing both sides of the first equation by the corresponding sides of the second one we obtain

\[
\frac{1 - x}{1 - X} = \frac{x}{X}
\]  

from which follows that $x=X$.

Substituting $x$ with $X$ in the first equation of (9) yields

\[
g^m \left[ \frac{1 + h^m}{1 - X} \right] = h^f \left[ \frac{1 + g^m}{1 + g^m} \right]
\]  

from which we get

\[
X = x = \frac{f h^f \left[ 1 + g^m \right]}{f h^f \left( 1 + g^m \right) + m h^m \left[ 1 + h^f \right]}.
\]  

When (12) holds the target vulnerability takes the form
3. Analysis of the minmax solution

When \( h = g = 1 \), \( x = X = \frac{f}{f + m} \) which means that the resources should be distributed between the detection and impact contests proportionally to the contest intensities. This resource distribution results in target vulnerability \( V = 0.25 \). Indeed, when \( h = g = 1 \) the effort ratio in each contest is equal to the relative resources ratio. Since the attacker and the defender distribute their resources equally the probability of the attacker’s success in both contests is \( w = p = 0.5 \), which produces the overall system survivability \( V = 0.25 \). The same vulnerability corresponds to the case of zero contest intensities (\( f = m = 0 \)), when the contest outcome does not depend on resources and is determined by chance.

The target vulnerabilities corresponding to the extreme values of contest intensities are presented in Table 1.

<table>
<thead>
<tr>
<th>( m, f )</th>
<th>( m, f )</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( g &lt; 1, h &lt; 1 )</td>
<td>( V = 0.25 )</td>
<td>( V = 0.5 )</td>
<td>( V = 0.5 )</td>
</tr>
<tr>
<td>( g &lt; 1, h &gt; 1 )</td>
<td>( V = 0.25 )</td>
<td>( V = 0.5 )</td>
<td>( V = 0 )</td>
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<tr>
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</tr>
<tr>
<td>( g = 1, h = 1 )</td>
<td>( V = 0.25 )</td>
<td>( V = 0.25 )</td>
<td>( V = 0.25 )</td>
</tr>
</tbody>
</table>

When \( h \neq 1 \) and \( g \neq 1 \) the optimal resource distribution \( x \) depends on these parameters. It increases with \( h \) and decreases with \( g \) (Fig. 1). Indeed, for the defender it is enough to succeed in any single contest in order to succeed in the bi-contest game. The growth of \( h \) gives advantage to the defender in the target detection contest and the defender allocates greater resource to this contest (\( x \) increases) in order to realize this advantage even by the price of lowering the chance to succeed in the impact contest. The growth of \( g \) gives advantage to the defender in the impact contest and the defender allocates greater resource to this contest (\( x \) decreases) in order to realize this advantage even by the price of lowering the chance to succeed in the target detection contest. As it can be seen from Fig. 1 the greater the corresponding contest intensity the greater the influence of \( h \) and \( g \) on the optimal resource distribution \( x \). It can be easily shown that in the case of zero intensity of one of contests, no resource should be allocated to this contest: \( x = 0 \) if \( f = 0 \) and \( x = 1 \) if \( m = 0 \). In the winner-take-all situation in one of contests all the resources should be allocated to this contest: \( x = 1 \) if \( f = \infty \) and \( x = 0 \) if \( m = \infty \).

In the winner-take-all situation in both contests (\( f = m = \infty \)) the defender should allocate all his resource to the contest where the effort ratio parameter is greater than 1: \( x = 1 \) if \( g < 1, h > 1 \); \( x = 0 \) if \( g > 1, h < 1 \). Any resource allocation \( x = 1 \) or \( x = 0 \) guarantees target survival if \( f > 1 \) and \( h > 1 \). If \( g < 1 \) and \( h < 1 \) no resource allocation can provide the target survival and \( V = 1 \).

When \( h \) approaches 0 allocating defender’s resource to the target detection contest is ineffective. Therefore \( x = 0 \) if \( h = 0 \) and \( g \neq 0 \). When \( g \) approaches 0 allocating defender’s
resource to the impact contest is ineffective. Therefore \( x = 1 \) if \( g = 0 \) and \( h \neq 0 \). On the contrary, when \( h \) or \( g \) approach infinity, the defender does not allocate all of his resources to the corresponding contest:

\[
\text{if } h = \infty \quad x = \frac{f \left( 1 + g^m \right)}{f \left( 1 + g^m \right) + mg^m};
\]

\[
\text{if } g = \infty \quad x = \frac{fh^f}{m(1 + h^f) + fh^f}.
\]

This is explained by the fact that when the corresponding contest intensity is low even achieving decisive superiority over adversary in allocated efforts does not guarantee success in the contest.

Figure 2 presents the system vulnerability as a function of effort ratio parameters \( h \) and \( g \). When \( h \) and \( g \) grow the vulnerability decreases as a result of defender's advantage in effort ratios. The greater the corresponding contest intensity the greater the influence of \( h \) and \( g \) on the resulting target vulnerability.

**Figure 1:** Optimal resource distribution \( x \) as function of effort ratio parameters for different values of contest intensities

**4. Resource allocation under uncertain contest intensities**

In many practical situations the exact values of the contest intensities cannot be determined. Therefore it would be useful to suggest a practical way to choose defender's resource distribution for certain intervals of contest intensities \( m \) and \( f \).
Figure 2: Target vulnerability as function of effort ratio parameters for different values of contest intensities

Let $X^*(x, m, f)$ be the attacker's resource distribution that maximizes the target vulnerability for the given $x$, $m$ and $f$. The most conservative defender's strategy is to choose the resource distribution $x^*$ that minimizes the maximal possible vulnerability in the range $f_{\text{min}} \leq f \leq f_{\text{max}}$, $m_{\text{min}} \leq m \leq m_{\text{max}}$ of contest intensities assuming that the attacker distributes his resource optimally:

$$\max_{f_{\text{min}} \leq f \leq f_{\text{max}}, \ m_{\text{min}} \leq m \leq m_{\text{max}}} V[X^*(x, m, f), x, m, f] \leq \max_{f_{\text{min}} \leq f \leq f_{\text{max}}, \ m_{\text{min}} \leq m \leq m_{\text{max}}} V[X^*(x, m, f), x, m, f] \text{for any } x \neq x^*. \quad (16)$$

The following numerical algorithm obtains the worst case target vulnerability $V^* = \max_{f_{\text{min}} \leq f \leq f_{\text{max}}, \ m_{\text{min}} \leq m \leq m_{\text{max}}} V[X^*(x, m, f), x, m, f]$ for any defender's resource distribution $x$ in the given range of contest intensities:

1. Assign $V^*=0$;
2. For $f_{\text{min}} \leq f \leq f_{\text{max}}$ and $m_{\text{min}} \leq m \leq m_{\text{max}}$ with step $\Delta f$ and $\Delta m$:  
   2.1. Find $X (0 \leq X \leq 1)$ that maximizes $V(X, x, m, f)$ (see Eq. (5)) applying any numerical procedure for finding maximum of a continuous function (Press et al. 1992) and assign: $\tilde{V}^* = \max_{X} V(X, x, m, f)$.
   2.3. if $V^* > \tilde{V}^*$ assign $V^* = \tilde{V}^*$.

The steps $\Delta f$ and $\Delta m$ should be chosen as a compromise between precision and computational complexity. In this work $\Delta f=0.01(f_{\text{max}}-f_{\text{min}})$ and $\Delta m=0.01(m_{\text{max}}-m_{\text{min}})$ were chosen.

Fig. 3. plots the worst case target vulnerability $V^*$ as function of $x$ for different ranges of the contest intensities (the corresponding ranges are presented in Table 2.) It can be
seen that the greater the contest intensities the greater sensitivity of $V^*$ to proper choice of defender's resource distribution $x$.

![Graph](image)

**Figure 3.** Worst case target vulnerability $V^*$ for different ranges of the contest intensities

<table>
<thead>
<tr>
<th></th>
<th>$f_{\text{min}}$</th>
<th>$f_{\text{max}}$</th>
<th>$m_{\text{min}}$</th>
<th>$m_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.1</td>
<td>0.5</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>5</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>C</td>
<td>0.1</td>
<td>0.5</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>D</td>
<td>2</td>
<td>5</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>E</td>
<td>12</td>
<td>15</td>
<td>12</td>
<td>15</td>
</tr>
</tbody>
</table>

Applying the presented numerical procedure for obtaining $V^*$ for any $x$ one can find the value of $x=x^*$ that minimizes $V^*$. Fig. 4 presents the example of optimal defender's resource distribution $x$ as a function of $g$ for the case of $h=1$ and $f=2$ for different fixed values of $m$ and for the range $2 \leq m \leq 8$.

![Graph](image)

**Figure 4:** $x^*$ as a function of $g$ for the case of $h=1$ and $f=2$.

It can be seen that when $g<1$, $x^*$ for $2 \leq m \leq 8$ coincides with $x^*$ for $m=8$ (in the case of low effort ratio parameter the highest contest intensity is worst for the defender), whereas when $g>1$, $x^*$ for $2 \leq m \leq 8$ coincides with $x^*$ for $m=2$ (in the case of high effort ratio parameter the lowest contest intensity is worst for the defender). The corresponding worst case target vulnerability is presented in Fig. 5. Again, when $g<1$ $V^*$ for $2 \leq m \leq 8$ coincides with $V^*$ for $m=8$, whereas when $g>1$ $V^*$ for $2 \leq m \leq 8$ coincides with $V^*$ for $m=2$. 
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Figure 5: \( V^*(x^*) \) as a function of \( g \) for the case of \( h=1 \) and \( f=2 \).

Figures 6 and 7 present the example of optimal defender's resource distribution \( x \) as a function of \( h \) for the case of \( g=1 \) and \( m=2 \) for different fixed values of \( f \) and for the range \( 2 \leq f \leq 8 \).

In analogy with the previous case when \( h<1 \), \( x^* \) and \( V^* \) for \( 2 \leq f \leq 8 \) coincides with \( x^* \) and \( V^* \) for \( f=8 \), whereas when \( h>1 \) \( x^* \) for \( 2 \leq f \leq 8 \) coincides with \( x^* \) for \( f=2 \).
This indicates that when finding the optimal strategy for the range of contest intensities, it is enough for the defender to analyze extreme values of the intensities.

5. Conclusion

The paper considers a bi-contest game between single attacker and single defender in which the attacker must win in both target detection and impact contests in order to destroy the target. Both agents distribute their constrained resources between the two contests. The defender distributes his resource trying to minimize the target vulnerability under assumption that for any defender's resource distribution the attacker achieves the greatest possible vulnerability by proper distribution of his resources.

It is shown that in the optimal minmax resource allocation the defender and the attacker should distribute their resources in the same proportion. The dependence of the optimal resource distribution on the contest intensities and on the effort ratio parameters is analyzed.

For the case of uncertain contest intensities a numerical algorithm is suggested that finds the defender's resource distribution minimizing the worst case target vulnerability.

References


For Biographic Sketch of the author, please see Guest Editorial